Differential Equations

UNIT- I

By

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Exact Differential Equation → A D.E. is said to be exact if it can be obtained from its primitive (without any change) by its differentiation.

Necessary condition → $Mdx + Ndy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Sufficient if " ", the D.E. is exact (other conditions may exist. for example, D.E. can be reduced to Exact D.E.)

For partial fractions to apply → Degree of Numerator is < Degree of denominator

Singular Solution → A solution of D.E. which cannot be obtained from its general solution

$y = a\cos x + b\sin x$ $\left[\frac{\partial y}{\partial a} = 0, \frac{\partial y}{\partial b} = 0 \right]$

Let $Mdx + Ndy = 0$, be an exact D.E., then gen. solⁿ is given by.

$$\int Mdx + \int \left\{ N - \int \frac{\partial}{\partial y} \int Mdx \right\} dy = C$$

Ex: $(x+y)^2 dx - (y^2 - 2xy - x^2) dy = 0$

On Comparing given D.E. with $Mdx + Ndy = 0$

$M = (x+y)^2$, $N = -(y^2 - 2xy - x^2)$

$$\frac{\partial M}{\partial y} = 2(x+y), \quad \frac{\partial N}{\partial x} = -(-2x-2y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M dx = \int (x+y)^2 dx = \frac{(x+y)^{2+1}}{(2+1)}$$

$$= \frac{(x+y)^3}{3}$$

[Differentiation of $(x+y)$ should be constant]

$$\int M dx + \int \left\{ -(y^2-2xy-x^2) - \frac{\partial}{\partial y}\int (x+y)^2 dx \right\} dy = c$$

$$\frac{(x+y)^3}{3} + \int \left\{ 2xy + x^2 - y^2 - \frac{3(x+y)^2}{3} \right\} dy = c$$

$$\frac{(x+y)^3}{3} + \int \left\{ 2xy + x^2 - y^2 - x^2 - y^2 - 2xy \right\}$$

$$\frac{(x+y)^3}{3} + \int (-2y^2)\, dy = c$$

$$\frac{(x+y)^3}{3} - \frac{2y^3}{3} = c$$

$$(x+y)^3 - 2y^3 = c'$$

13/12/19 Solving Exact D.E. when it is in non-exact form

① Inspection Method

$$\frac{-1}{xy} + \log\left|\frac{y}{x}\right| = c \quad \left\{ \quad \frac{-1}{xy} + \log\left(\frac{y}{x}\right) = c \right.$$

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(2) Homogeneous function of 'n' order

If $f(kx, ky) = k^n f(x,y)$, then (x,y) is homogeneous.

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$ Both are homogeneous functions of same order

If $Mx + Ny \neq 0$, then I.F. $= \frac{1}{Mx+Ny}$

$\Rightarrow$ If $Mx + Ny = 0$, $\frac{M}{N} = \frac{-y}{x}$

Then $Mdx + Ndy = 0$

$$\frac{M}{N}dx + dy = 0$$

$$\left(\frac{-y}{x}\right)dx + dy = 0$$

$$-\frac{dx}{x} + \frac{dy}{y} = 0$$

Example- $(x^4 + y^4)dx - xy^3 dy = 0$ --- (1)

On comparing the given D.E. with $Mdx + Ndy = 0$

$$M = x^4 + y^4, \quad N = -xy^3$$

$$\frac{\partial M}{\partial y} = 4y^3, \quad \frac{\partial N}{\partial x} = -y^3$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ The given D.E. is not exact.

But it is homogeneous D.E.

Now we find $Mx + Ny = (x^4 + y^4)x + (-xy^3)y$

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$= x^5 \neq 0 \quad \therefore \text{I.F.} = \dfrac{1}{x^5}$

Multiply the given D.E (1) by with $\dfrac{1}{x^5}$, we get

$$\left(\frac{x^4+y^4}{x^5}\right)dx - \frac{xy^3}{x^5}dy = 0 \quad \text{---(2)}$$

Comparing Deq. (2) with $Mdx + Ndy = 0$

$$M = \frac{1}{x} + \frac{y^4}{x^5}, \quad N = -\frac{y^3}{x^4}$$

$$\frac{\partial M}{\partial y} = \frac{4y^3}{x^5}, \quad \frac{\partial N}{\partial x} = \frac{4y^3}{x^5}$$

Clearly $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Now the given D.E. is exact and the solution is given by –

$$\int M dx + \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy = c$$

$$\int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx + \int\left\{\frac{-y^3}{x^4} - \frac{\partial}{\partial y}\int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx\right\}dy = c$$

$$\log x - \frac{y^4}{4x^4} + \int\left\{\frac{-y^3}{x^4} - \frac{\partial}{\partial y}\left(\log x - \frac{y^4}{4x^4}\right)\right\}dy = c$$

$$\log x - \frac{y^4}{4x^4} + \int\left(\frac{-y^3}{x^4} - \cancel{\frac{1}{x}} + \frac{y^3}{x^4}\right)dy = c$$

$\log x - \dfrac{y^4}{4x^4} = c$ which is the required G.S. of given D.E.

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Solve : $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0$ --- (1)

⇒ On comparing given D.E. with $Mdx+Ndy=0$, we have,

$M=x^2y-2xy^2$

$N=-(x^3-3x^2y)$

$$\frac{\partial M}{\partial y}=x^2-4xy$$

$$\frac{\partial N}{\partial x}=-3x^2+6xy$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

∴ The given D.E is not exact.

Since the sum of powers of variables in each term of DE is same, it is homogeneous D.E.

$$\therefore\quad Mx+Ny=x^3y-2x^2y^2-x^3y+3x^2y^2$$
$$=x^2y^2$$

$$I.F.=\frac{1}{x^2y^2}$$

Multiply the given D.E. by $\frac{1}{x^2y^2}$,

$=c$

we have

$$\frac{(x^2y-2xy^2)\,dx}{x^2y^2}-\frac{(x^3-3x^2y)\,dy}{x^2y^2}=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0 \quad ---(2)$$

Now $M=\frac{1}{y}-\frac{2}{x}$, $N=-\left(\frac{x}{y^2}-\frac{3}{y}\right)$

Now $\frac{\partial M}{\partial y}=\frac{-1}{y^2}$, $\frac{\partial N}{\partial x}=\frac{-1}{y^2}$

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$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ D.E. is exact.

General solution of the given D.E ① is given by

$$\int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx \right) dy = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x} \right) dx + \int \left(\frac{-x}{y^2} + \frac{3}{y} \right) - \frac{\partial}{\partial y} \int \left(\frac{1}{y} - \frac{2}{x} \right) dy = c$$

$$\frac{x}{y} - 2\log x + \int \left\{ \left(-\frac{x}{y^2} + \frac{3}{y} - \frac{\partial}{\partial y} \left(\frac{x}{y} - \log x \right) \right\} dy = c$$

$$\frac{x}{y} - 2\log x + \int \left(\frac{-x}{y^2} + \frac{3}{y} - \frac{x}{y^2} \right) dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

which is the required G.S. of given D.E ①.

Rule III $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$Mx - Ny \neq 0$

$$I.F. = \frac{1}{Mx - Ny}$$

If $Mx - Ny = 0$

$$\frac{M}{N} = \frac{y}{x}$$

Solve: $(1+xy)y\,dx + (1-xy)x\,dy = 0$ --- (1)

On comparing the given D.E. with $M\,dx + N\,dy = 0$,

$M = (1+xy)y \quad , N = (1-xy)x$

$\frac{\partial M}{\partial y} = 1 + 2xy \quad , \frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ The given D.E. is not exact.

Now, $Mx - Ny = (1+xy)xy - (1-xy)xy$
$= xy + x^2y^2 - xy + x^2y^2$
$= 2x^2y^2$

$I.F. = \frac{1}{2x^2y^2}$

Now multiply (1) with $\frac{1}{x^2y^2}$, we have

$\frac{(1+xy)}{2x^2y^2} y\,dx + \frac{(1-xy)x}{2x^2y^2} dy = 0$

$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0$

Now $M = \frac{1}{2x^2y} + \frac{1}{2x}$, $N = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{d(\log x)}{dx} = \frac{1}{x}$

$\therefore$ General solⁿ of given DE is given by:

$$\int M dx + \int \left(N - \frac{\partial}{\partial y}\int M dx\right) dy = c$$

$$\cancel{xy + \frac{x^2y^2}{2}} \quad \frac{-1}{2xy} + \frac{1}{2}\log x + \int \left(\frac{1}{2xy^2} - \frac{1}{2y} - \left(\frac{1}{2xy^2}\right)\right) dy = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c'$$

Rule-IV $\quad M dx + N dy = 0$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x or constant.

then I.F. $= e^{\int f(x) dx}$

Rule-V $\quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y or constant

$\therefore$ I.F. $= e^{\int f(y) dy}$

Rule-VI If D.E. is of the form:

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

I.F. $= x^h y^k$

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Simultaneous D.E $\rightarrow$ The D.E. in which No. of dependent variables = No. of L.D.E.

Solve $\frac{dx}{dt} - 7x + y = 0$

(1) $\frac{dy}{dt} - 2x - 5y = 0$

$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

(2)

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

Let $D \equiv \frac{d}{dt}$ (defined as), then given simultaneous system of eq^n can be written as

$Dx + (D-2)y = 2\cos t - 7\sin t$ --- (1)

$(D+2)x - Dy = 4\cos t - 3\sin t$ --- (2)

~~Multiply~~ Operating eq. (1) with D and eq (2) with (D-2), we get

$D^2x + D(D-2)y = D(2\cos t - 7\sin t)$ --- (3)

$(D-2)(D+2)x - D(D-2)y = (D-2)(4\cos t - 3\sin t)$ --- (4)

Adding eq. (3) and (4), we get

$D^2x + (D^2-4)x = -2\sin t - 7\cos t + 4\sin t - 3\cos t$

$-8\cos t + 6\sin t$

$\Rightarrow (D^2+D^2-4)x = -18\cos t$

$(D^2-2)x = -9\cos t$

This is LDE with constant coefficients

AE is $m^2-2=0$

$m = \pm\sqrt{2}$

$CF = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t}$ where c_1 & c_2 are arbitrary constants

$P.I. = \frac{1}{(D^2-2)}(-9\cos t) = 3\cos t$

$G.S. = CF + P.I.$

$x(t) = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t$

Now adding eq. ① and ②, we get

$Dx + (D+2)x - 2y = 6\cos t - 10\sin t$

$2y = 2(Dx + x - 3\cos t + 5\sin t)$

$y = Dx + x - 3\cos t + 5\sin t$

$y = \sqrt{2}c_1e^{\sqrt{2}t} - \sqrt{2}c_2e^{-\sqrt{2}t} - 3\sin t + c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t - 3\cos t + 5\sin t$

$y(t) = (1+\sqrt{2})c_1e^{\sqrt{2}t} + (1-\sqrt{2})c_2e^{-\sqrt{2}t} + 2\sin t$

③ Solve: $t\frac{dx}{dt} + y = 0$ --- ①

$t\frac{dy}{dt} + x = 0$ --- ②

Putting the value of x from (2) in (1), we get

$$t\frac{d}{dt}\left(-t\frac{dy}{dt}\right)+y=0$$

$$-t\left[\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right]+y=0$$

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=0 \quad \text{--- (3)}$$

This is a homogeneous linear D.E.

For Transforming eq. (3) into a linear equation with constant coefficients, by changing independent variable t to z using substitution,

$$t=e^z.$$

$$z=\log t$$

$$\frac{dz}{dt}=\frac{1}{t}$$

$$\therefore \frac{dy}{dt}=\frac{dy}{dz}\cdot\frac{dz}{dt}=\frac{1}{t}\frac{dy}{dz}$$

$$t\frac{dy}{dt}=\frac{dy}{dz} \text{ --- (4) and } t^2\frac{d^2y}{dt^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz} \text{ --- (5)}$$

Now if we take $\frac{d}{dz}\equiv D$.

we can write (4) & (5) as

$$t\frac{dy}{dt}=Dy$$

$$t^2\frac{d^2y}{dt^2}=D(D-1)y$$

$[D(D-1)+D-1]y=0$

$(D^2-1)y=0$

AE of the transformed eq^n is $m^2-1=0$

$m=\pm 1$

$C.F. = c_1e^z + c_2e^{-z} = c_1t + c_2t^{-1}$

$P.I. = 0$

$y = C.F. + P.I. = c_1t + c_2t^{-1}$ ---- (6)

Using (6) in (2), we get

$t\dfrac{d(c_1t + c_2t^{-1})}{dt} + x = 0$

$t\left[c_1 - \dfrac{c_2}{t^2}\right] + x = 0$

$x = t\left[\dfrac{c_2}{t^2} - c_1\right] = c_2t^{-1} - c_1t$ ---- (7)

(6) and (7) are the required solutions of the given simultaneous DE.

(1) Let $D \equiv \frac{d}{dt}$, then given simultaneous D.E can be written as :-

$(D-7)x + y = 0$ ---- (1)

$-2x + (D-5)y = 0$ ---- (2)

Multiply eq. (1) with $(D-5)$ and subtract with eq. (2), we get

$(D-7)(D-5)x + (D-5)y + 2x - (D-5)y = 0$

$(D^2-12D+35+2)x = 0$

$(D^2-12D+37)x = 0$

A.E. is $m^2 - 12m + 37 = 0$

$$m = \frac{-(-12) \pm \sqrt{144 - 148}}{2(1)}$$

$$= \frac{12 \pm 2i}{2} = 6 \pm i$$

$$C.F. = e^{6t}\left[c_1 \cos t + c_2 \sin t\right]$$

$$P.I. = 0$$

$$\therefore x(t) = C.F. + P.I. = e^{6t}(c_1 \cos t + c_2 \sin t) \quad \text{---(3)}$$

Putting value of x from (3) in eq (1), we get

$$(D-7)\left(e^{6t}c_1 \cos t + e^{6t}c_2 \sin t\right) + y = 0$$

$$\left[6e^{6t}c_1 \cos t + e^{6t}c_1(-\sin t) + 6e^{6t}c_2 \sin t + e^{6t}c_2 \cos t - 7e^{6t}c_1 \cos t - 7e^{6t}c_2 \sin t + y = 0\right.$$

$$y = e^{6t}c_1 \cos t + e^{6t}c_2 \sin t + e^{6t}c_1 \sin t - e^{6t}c_2 \cos t$$

$$y = c_1\left(e^{6t}\cos t + e^{6t}\sin t\right) + c_2\left(e^{6t}\sin t - e^{6t}\cos t\right)$$

$$= e^{6t}\left[(c_1 + c_2)\sin t + (c_1 - c_2)\cos t\right] \quad \text{---(4)}$$

Eq. (3) and (4) are the required solutions of the given simultaneous D.E.

[Sol^n must be independent]

① Solve: $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$

By taking first two ratios,

$$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx}$$

$$a(c-a)x\,dx = b(b-c)y\,dy$$

Integrating both sides, we get

$$a(c-a)x^2 - b(b-c)y^2 = c_1 \quad \text{--- ①}$$

Now taking multipliers x, y, z, we have

$$\frac{c\,dz}{(a-b)xy} = \frac{ax\,dx + by\,dy + cz\,dz}{xyz(b-c+c-a+a-b)}$$

$$\frac{c\,dz}{(a-b)xy} = \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

$$ax\,dx + by\,dy + cz\,dz = 0$$

Integrating both sides, we have

$$\frac{ax^2}{2} + \frac{by^2}{2} + \frac{cz^2}{2} = c_2$$

$$ax^2 + by^2 + cz^2 = c_2' \quad \text{--- ②}$$

$\phi(c_1, c_2')$ is the required complete solution of the given simultaneous differential equations.

(2) $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$

By taking first two relations, we have

$\frac{dx}{1} = \frac{dy}{3}$

$3dx = dy$

$dy - 3dx = 0$

Integrating both sides,

$y - 3x = c_1$ --- (1) c_1 is arbitrary constant.

Again by taking last two relations, we have

$\frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$

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$\int \frac{dy}{3} = \frac{1}{5}\int \frac{5dz}{5z+\tan c_1}$

Integrating both sides.

$\frac{y}{3} = \frac{1}{5}\log|5z+\tan c_1| + c_2$ c_2 is arbitrary constant

$y - \frac{3}{5}\log|5z+\tan(y-3x)| = c_2'$ where $c_2' = 3c_2$

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$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0 y = Q(x)$

Put $x = e^z \Rightarrow z = \log x$

$x^n D^n = D'(D'-1)(D'-2)\dots(D'-\overline{n-1})$

$xD = D'$

$x^2D^2 = D'(D'-1)$ where $D' \equiv \frac{d}{dz}$

which is L.D.E. with constant coefficients which can be solved.

Reducible ~~to~~ homogenous L.D.E.

$$A_n(a+bx)^n \frac{d^n y}{dx^n} + A_{n-1}(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0 y = Q(x)$$

Put $(a+bx) = t \Rightarrow b = \frac{dt}{dx}$

Now $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = b\cdot\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b\,dy}{dt}\right)$$

$$= b\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= b\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dt^2}$$

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$$\frac{d^n y}{dx^n} = b^n\frac{d^n y}{dt^n}$$

$$A_n t^n b^n \frac{d^n y}{dt^n} + A_{n-1} t^{n-1} b^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + A_0$$

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Unit-I

Total Differential Equations (Pfaffian eqn)

The equation of the form $\sum_{i=1}^{n} \phi_i(x_1, x_2, x_3, \ldots, x_n) dx_i = 0$

$$\phi_1(x_1, x_2, \ldots, x_n) dx_1 + \phi_2(x_1, x_2, \ldots, x_n) dx_2 + \ldots + \phi_n(x_1, x_2, \ldots, x_n) dx_n = 0$$ in which

ϕ_i ; $i = 1, 2, 3, \ldots, n$ are in general functions of independent variables $x_1, x_2, \ldots, x_n$ is called total D.E. or Pfaffian D.E.

Total derivative of $f(x,y,z)$ is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Necessary and sufficient condition for the total D.E.:-

$$Pdx + Qdy + Rdz = 0.$$

The necessary and sufficient condition for the total D.E. $Pdx + Qdy + Rdz = 0$ to be integrable is

$$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

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$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

Necessary condition:

If total D.E. $Pdx+Qdy+Rdz=0$ is integrable then

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

Sufficient condition,

Let $Pdx+Qdy+Rdz=0$ --- (1) and

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial R}{\partial x}\right)=0$$

then (1) is integrable.

Method of Inspection:-

Ex-1 Solve:- $xdy-ydx+2x^2zdz=0$ --- (1)

On comparing (1) with $Pdx+Qdy+Rdz=0$

$$P=-y,\quad Q=x,\quad R=2x^2z$$

Now $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$$=-y(0-0)+x(4xz-0)+2x^2z(-1-1)$$

$= 4x^2z - 4x^2z = 0$

$\therefore$ (1) is integrable.

Now, (1) can be written as

$$\frac{x\,dy - y\,dx}{x^2} + 2z\,dz = 0$$

$$d\left(\frac{y}{x}\right) + 2z\,dz = 0$$

Integrating on both sides,

$\frac{y}{x} + z^2 = c$, where c is arbitrary constant

which is the required solution of given differential equation.

Ex-2 Solve:- $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2z\,dz = 0$ --- (1)

Comparing (1) with $P\,dx + Q\,dy + R\,dz = 0$,

$P = 2x^2 + 2xy + 2xz^2 + 1$
$Q = 1$
$R = 2z$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1(0 - 4xz) +$

$2z(2x - 0) = -4xz + 4xz = 0$

$= 4x^2z - 4x^2z = 0$

$\therefore$ (1) is integrable.

Now, (1) can be written as

$$\frac{x\,dy - y\,dx}{x^2} + 2z\,dz = 0$$

$$d\left(\frac{y}{x}\right) + 2z\,dz = 0$$

Integrating on both sides,

$$\frac{y}{x} + z^2 = c$$, where c is arbitrary constant

which is the required solution of given differential equation.

Ex-2 Solve :- $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2z\,dz = 0$ --- (1)

Comparing (1) with $P\,dx + Q\,dy + R\,dz = 0$,

$P = 2x^2 + 2xy + 2xz^2 + 1$
$Q = 1$
$R = 2z$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1(0 - 4xz) + 2z(2x - 0) = -4xz + 4xz = 0$

$\therefore$ (1) is integrable and can be written as

$$2x(x+y+z^2)dx + dx + dy + 2z\,dz = 0 \quad \text{--- (2)}$$

Dividing the eq. (2) throughout by $x+y+z^2$, we get

$$2x\,dx + \frac{dx+dy+2z\,dz}{x+y+z^2} = 0$$

Integrating on both sides, we have

$$x^2 + \log(x+y+z^2) = c$$ where c is arbitrary constant

which is required solⁿ of (1).

Ex 3 $(y^2+z^2-x^2)dx - 2xy\,dy - 2xz\,dz = 0$ --- (1)

Ex-4 ~~On comparing (1)~~ $\cancel{(2x^2y+2xy^2+2xyz+1)dx + (x^3+x^2y+x^2z+2xyz+2y^2z+2yz^2+1)dy + (xy^2+y^3+y^2z+1)dz = 0}$

Comparing (1) with $Pdx + Qdy + Rdz = 0$,

$P = y^2+z^2-x^2$, $Q = -2xy$, $R = -2xz$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$(y^2+z^2-x^2)(0-0) + (-2xy)(-2z-2z) + (-2xz)(2y+2y)$

$= 4xyz + 4xyz - 8xyz = 0$

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$\therefore$ (1) is ~~dif~~ integrable and can be written as

$$(x^2+y^2+z^2)\,dx - 2x^2 dx = 2xy\,dy + 2xz\,dz$$

$$(x^2+y^2+z^2)\,dx = 2x(x\,dx + y\,dy + z\,dz)$$

$$\frac{dx}{x} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$$

Integrating on both sides,

$$\log x + \log c = \log(x^2+y^2+z^2)$$

$$x^2+y^2+z^2 = xc$$

which is the required solution of (1)

Ex-4 On comp. given DE (1) with $P\,dx + Q\,dy + R\,dz = 0$
we have $P=$, $Q=$, $R=0$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$(2x^2y + 2xy^2 + 2xyz + 1)(x^2 + 2xy + 2y^2 + 4yz - 2xy - 3y^2 - 2yz)$$

$$+ (x^3 + x^2y + x^2z + 2xyz + 2y^2z + 2yz^2 + 1)(y^2 - 2xy)$$

$$+ (xy^2 + y^3 + y^2z + 1)(2x^2 + 4xy + 2xz - 3x^2 - 2xy - 2xz - 2yz)$$

$$= (x^2 - y^2 + 2yz + 2x^4y - 2x^2y^3 + 4x^2y^2z + 2x^3y^2$$
$$- 2xy^4 + 4xy^3z + 2x^3yz - 2xy^3z + 4xy^2z^2)$$
$$+ (x^2y^3 - x^3y^2 - 3x^2y^2z - 2xy^3z + y^2 - 2xy$$
$$+ 2y^4z + 2y^3z^2 - 2x^4y - 2x^3yz - 4xy^2z^2)$$
$$+ (-x^3y^2 + x^2y^3 - x^2y^2z - x^2 + 2xy^4 + 2xy$$
$$- 2y^4z - 2y^3z^2 - 2yz) =$$

$$= x^2 - x^2 - y^2 + y^2 + 2yz - 2yz + 2x^4y - 2x^4y$$
$$- 2x^2y^3 + x^2y^3 + x^2y^3 + 4x^2y^2z - 3x^2y^2z$$
$$- x^2y^2z + 2x^3y^2 - x^3y^2 - x^3y^2 - 2xy^4$$
$$+ 2xy^4 + 4xy^3z - 2xy^3z - 2xy^3z +$$
$$2x^3yz - 2x^3yz + 4xy^2z^2 - 4xy^2z^2 + 2y^3z^2 - 2y^3z^2$$
$$+ 2y^4z - 2y^4z - 2xy + 2xy$$

$$= 0$$

$\therefore$ (1) is integrable and can be written as:

$$[2xy(x+y+z) + 1]\,dx + [x^2(x+y+z) + 2yz(x+y+z) + 1]\,dy$$
$$+ [y^2(x+y+z) + 1]\,dz = 0$$

$$2xy(x+y+z)\,dx + (x^2+2yz)(x+y+z)\,dy +$$
$$y^2(x+y+z)\,dz + dx + dy + dz = 0$$

Dividing throughout by $(x+y+z)$, we have

$$2xy\,dx + (x^2+2yz)\,dy + y^2dz + \frac{dx+dy+dz}{x+y+z} = 0$$

$$(2xy\,dx + x^2dy) + (2yz\,dy + y^2dz) + \frac{dx+dy+dz}{x+y+z} = 0$$

$$d(x^2y) + d(y^2z) + \frac{dx+dx+dy}{x+y+z} = 0$$

Integrating both sides,

$x^2y + y^2z + \log(x+y+z) = c$ which is the required solution of (1).

Method for solution of homogenous equations:-

Let $P\,dx + Q\,dy + R\,dz = 0$ --- (1) be an integrable equation in which P, Q, R are homogenous functions of same degree n. In such case, one variable is separated from the other two by the substitution:

$$x = uz, \quad y = vz.$$

$$dx = u\,dz + z\,du, \quad dy = v\,dz + z\,dv$$

Putting values of x, y, dx and dy in (1), we have

$$z^{n+1}\{f(u,v)\,du + g(u,v)\,dv\} + z^n\{u f(u,v) + v\,g(u,v) + h(u,v)\}\,dz = 0$$

Two cases → (i) coefficient of dz is zero. We will have eqn in two variables u and v which can be rearranged and integrated.

(ii) If coefficient not zero, z can be separated from u & v.

Now two cases arise.

(1) If $Px + Qy + Rz = 0$, then substitute $x = uz$ and $y = vz$. so that $dx = zdu + udz$ and $dy = vdz + zdv$

(2) If $Px + Qy + Rz \neq 0$, then I.F. $= \dfrac{1}{Px + Qy + Rz}$

Multiply (1) with I.F. and then solve. $\boxed{\text{Rearrange}}$

Solve :- $yz^2(x^2 - yz)\,dx + x^2z(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ --- (1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$

$P = yz^2(x^2 - yz)$, $R = xy^2(z^2 - xy)$

$Q = x^2z(y^2 - xz)$

Now $P\left(\dfrac{\partial Q}{\partial z} - \dfrac{\partial R}{\partial y}\right) + Q\left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right) + R\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right)$

$= yz^2(x^2 - yz)(x^2y^2 - 2xz^2y - 3x^2y^2)$

$+ x^2z(y^2 - xz)(y^2z^2 - 2xy) - 2x^2yz +$

$Px + Qy + Rz = 0$

dz will eliminate.

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The given total DE is integrable and homogeneous

Again Now $Px + Qy + Rz = xyz^2(x^2 - yz) + x^2yz(y^2 - xz) + xy^2z(z^2 - xy) =$

$= x^3yz^2 - xy^2z^3 + x^2y^3z - x^3yz^2 + xy^2z^3 - x^2y^3z$

$= 0$

Put $x = uz$, $y = vz$ $\quad dx = udz + zdu$, $dy = vdz + zdv$

Then substitute $x = uz$ and $y = vz$, dx and dy in ①,

$vz^3(u^2z^2 - vz^2)[udz + zdu] + u^2z^3(v^2z^2 - uz^2)[vdz + zdv] + uv^2z^3(z^2 - uvz^2)dz$

$\Rightarrow z^5v(u^2 - v)(udz + zdu) + z^5u^2(v^2 - u)(vdz + zdv) + z^5uv^2(1 - uv)dz = 0$

z^5 goes to RHS and dz terms cancel each other

$\Rightarrow v(u^2 - v).du + u^2(v^2 - u)dv = 0$

$(u^2v - v^2)du + (u^2v^2 - u^3)dv = 0$

Dividing by u^2v^2, we have

$\frac{1}{v}du - \frac{1}{u^2}du + dv - \frac{u}{v^2}dv = 0$

$$\left(\frac{1}{v}du - \frac{u}{v^2}dv\right) - \frac{1}{u^2}du + dv = 0$$

$$d\left(\frac{u}{v}\right) - \frac{1}{u^2}du + dv = 0$$

Integrating both sides, we have

$$\frac{u}{v} + \frac{1}{u} + v = c \text{ where } c \text{ is arbitrary constant.}$$

$$u^2 + v + uv^2 = cuv$$

Putting values of u and v, we have

$$\frac{x^2}{z^2} + \frac{y}{z} + \frac{x}{z}\left(\frac{y^2}{z^2}\right) = \frac{cxy}{z^2}$$

$$zx^2 + yz^2 + xy^2 = cxyz$$

which is the required solution of (1).

Solve :- $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$ --- (1)

Comparing (1) with standard form $Pdx + Qdy + Rdz = 0$,

$$P = y^2+yz, \quad Q = xz+z^2, \quad R = y^2-xy$$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (y^2+yz)(x+2z-2y+x) + (xz+z^2)(-y-y) +$$

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$$(y^2-xy)(2y+z)-z)$$

$$= 2(y^2+yz)(x-y+z) + (-2y)(xz+z^2) + 2y(y^2-xy)$$

$$= 2(\cancel{xy^2} - \cancel{y^3} + \cancel{zy^2} + \cancel{xyz} - \cancel{y^2z} + \cancel{yz^2}) - \cancel{2xyz} - \cancel{2yz^2} + \cancel{2y^3} - \cancel{2xy^2} = 0$$

$\therefore$ (1) is integrable and homogeneous.

$$Px+Qy+Rz = xy^2+xyz+xyz+yz^2+y^2z-\cancel{xyz}$$

$$= xy(y+z) + yz(y+z)$$

$$= (xy+yz)(y+z) = y(x+z)(y+z) = I \text{ (say)} \quad \text{--- (2)}$$

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$\cancel{dI =}$

Now multiplying (1) by $\frac{1}{I}$, we have

$$\frac{(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2-xy)\,dz}{y(x+z)(y+z)} = 0 \quad \text{--- (3)}$$

$$dI = (x+z)(y+z)\,dy + y(y+z)(dx+dz) + y(x+z)(dy+dz)$$

$$dI = (x+z)(y+z)\,dy + y(y+z)\,dx + y(x+z)\,dy + [y(y+z)+(x+z)]\,dz \quad \text{--- (4)}$$

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Again, now by (3),

$$\frac{y(y+z)dx + z(x+z)dy + y(y-x)dz}{I} = 0 \quad \text{--- (5)}$$

~~Using~~ (4) can be written as

$$dI - y(x+z)dy - y(x+z)dy - y\,(\cancel{y}+z)\cancel{dz}$$
$$- y(x+z)dz$$
$$= y(y+z)dx + z(x+z)dy$$
--- (6)

Using (6) in (5),

$$\frac{dI + [-yx - yz - yx - yz]dy + [-y^2 \cancel{-yz} - yx - yz + y^2 - xy]dz}{I}$$

$$\frac{dI}{I} \frac{-2y(x+z)dy - 2y(x+z)dz}{I} = 0$$

$$\frac{dI}{I} - \frac{2y(x+z)(dy+dz)}{I} = 0$$

$$\frac{dI}{I} - \frac{2y(x+z)(dy+dz)}{y(x+z)(y+z)} = 0$$

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$$\frac{dI}{I} - 2\frac{(dy+dz)}{y+z} = 0$$

Integrating both sides, $\log I - 2\log(y+z) = \log c$

$$\frac{I}{(y+z)^2} = c$$

$$y(x+z)\cancel{(y+z)} = c(y+z)^{\cancel{2}}$$

$$y(x+z) = c(y+z)$$

which is the required solution of given T.D.E.

Solve:- $(yz+z^2)dx - xzdy + xydz = 0$

Method of Auxiliary equations:-

Consider the total differential equation

$Pdx + Qdy + Rdz = 0$ — ① for which the condition of integrability is -

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad \text{--- ②}$$

On comparing ① and ②, we have simultaneous equations known as Auxiliary equations.

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)} \quad \text{--- ③}$$

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Let $u = a$ and $v = b$ be their integrals. Then we will find out the value of two functions A and B such that the eq. (1) becomes identical with

$$A\,du + B\,dv = 0 \quad \text{---} (4)$$

Comparing (4) with (1), find values of A and B. Put these values of A and B in (4)

and integrate the D.E. It will give the required solution when values of u and v are substituted in the relation obtained after integration.

⇒ This method fails when :-

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0}$$

i.e. $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

OR

Curl X = 0 . $\nabla \times \vec{F} = \text{Curl } F$

(*) Comparing the given total differential equation with $P\,dx + Q\,dy + R\,dz = 0$, we get

$$P = yz + z^2, \; Q = -xz, \; R = xy$$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (yz + z^2)(-x - x) + (-xz)(y - y - 2z) + xy(z + z)$$

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$= -2xyz - 2xz^2 - \cancel{2xyz} + 2xz^2 + 2xyz = 0$

$\therefore$ (1) is integrable and (1) is also homogeneous

$$Px + Qy + Rz = (yz+z^2)x + (-xz)(y) + (xy)(z)$$

$$= xyz + xz^2$$

which is not equal to zero.

Let $xyz + xz^2 = I$ --- (2)

Then, $I.F. = \dfrac{1}{Px+Qy+Rz} = \dfrac{1}{xyz+xz^2}$

$\therefore$

Multiplying (1) with I.F., we have

$$\frac{(yz+z^2)\,dx - xz\,dy + xy\,dz}{x(yz+z^2)} = 0 \quad \text{--- (3)}$$

From (2), total derivative of I is

$$dI_1 = (yz+z^2)\,dx + xz\,dy + 2xz\,dz + xy\,dz$$

$$(yz+z^2)\,dx = dI - xz\,dy - (2xz+xy)\,dz$$

and $xyz + xz^2 = I$

Substituting value of $(yz+z^2)\,dx$ in (3),

$$\frac{dI - xz\,dy - 2xz\,dz - xy\,dz - xz\,dy + xy\,dz}{I}$$

$$\frac{dI}{I} - \frac{2xz\,dy + 2xz\,dz}{xz(y+z)} = 0$$

$$\frac{dI}{I} - \frac{2(dy + dz)}{y+z} = 0$$

Integrating on both sides, we get

$$\log I - 2\log(y+z) = \log c$$

$$\log(xyz + xz^2) - \log(y+z)^2 = \log c$$

$$\log\left[\frac{xyz + xz^2}{(y+z)^2}\right] = \log c$$

$$\frac{xyz + xz^2}{(y+z)^2} = c$$

$$\frac{xz(y+z)}{(y+z)^2} = c \Rightarrow xz = c(y+z)$$

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$$3x^2dx + 3y^2dy - (x^3+y^3+e^{2z})dz = 0 \quad \text{--- (1)}$$

On comparing with $Pdx + Qdy + Rdz = 0$, we have

$$P = 3x^2, \quad Q = 3y^2, \quad R = -(x^3+y^3+e^{2z})$$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) =$

$$= 3x^2(0 + 3y^2) + 3y^2(-3x^2 - 0) + (-x^3 - y^3 - e^{2z})(0 - 0)$$

$$= 9x^2y^2 - 9x^2y^2 = 0$$

$\therefore$ (1) is integrable.

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A.E of the given eqⁿ (1) are

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

$$\frac{dx}{3y^2} = \frac{dy}{-3x^2} = \frac{dz}{0} \quad \text{--- (2)}$$

By taking last two ratios,

$$dz = 0$$

Integrating both sides.

$$z = c_1 = v \text{ (say)} \quad \text{--- (3a)}$$

By taking first two ratios, we have

$$-x^2dx = y^2dy \Rightarrow x^2dx + y^2dy = 0$$

Integrating both sides.

$$x^3 + y^3 = c_2 = u \text{ (say)} \quad \text{--- (3b)}$$

Now $A\,du + B\,dv = 0$ --- (4)

$$A(3x^2dx + 3y^2dy) + B\,dz = 0$$

$$3A\,x^2dx + 3Ay^2dy + B\,dz = 0 \quad \text{--- (5)}$$

On comparing (1) and (5),

$$A = 1,\ B = -(x^3+y^3+e^{2z})$$

$$\Rightarrow A = 1,\ B = -(u+e^{2v})$$

Putting values of A and B in (4),

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(1) $du - (u + e^{2v})\,dv = 0$

$du - u\,dv - e^{2v}dv = 0$

$$\frac{du}{dv} - u = e^{2v} \quad \text{---} (6)$$

This is linear D.E. in v.

Its solution is given by $\quad I.F. = e^{-\int dv} = e^{-v}$

$$u \cdot e^{-v} = \int e^{2v} \cdot e^{-v} dv + c_3$$

$$u \cdot e^{-v} = e^{v} + c_3$$

$$u = e^{2v} + c_3 e^{v} \quad \text{---} (7)$$

Putting values of u and v in (7) from (3a) & (3b)

$$x^3 + y^3 = e^{2z} + c_3 e^{z}$$

which is the required solution of T.D.E.

required solution of given total differential equation

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General Method :-

Consider the total D.E.,

$$Pdx + Qdy + Rdz = 0 \quad \text{---} \; (1)$$

and satisfy the integrability condition.

Sol:- (i) Take $z =$ constant, then (1) reduces to

$$Pdx + Qdy = 0 \quad \text{---} \; (2)$$

(ii) Now integrate (2) considering z as constant, then we have-

$$U(x, y) = f(z) \quad \text{---} \; (3)$$

where $f(z)$ is constant w.r.t. x and y
arbitrary func^n and

(iii) Differentiating ③ by taking z as variable also and then comparing it with (1) and we will get relationship b/w df and dz.

(iv) If the coefficients of df and dz involves func" of x and y, it would be possible to eliminate them with the help of relation ③.

(v) Thus, we shall get an equation in df and dz which will be independent of x and y.

(vi) The value of f(z) will be obtained by integrating the above equation, which when substituted in ③, will give the complete solution.

Solve: $3x^2dx + 3y^2dy - (x^3+y^3+e^{2z})dz = 0$ --- ①

On comp. given T.D.E. with standard form,

$P = 3x^2, \quad Q = 3y^2, \quad R = -(x^3+y^3+e^{2z})$

① satisfies the integratibility condition.

Let z = constant $\Rightarrow dz = 0$

Then, (1) becomes $3x^2dx + 3y^2dy = 0$ --- ⊛

Integrating both sides of ⊛ treating z as constant.

$x^3 + y^3 = f(z)$ --- ②

Now, differentiating ② partially w.r.t. x, y, z considering z as a variable, we have

$$3x^2dx + 3y^2dy - f'(z)\,dz = 0 \quad \text{--- ③}$$

On comparing ① and ③, we have

$$x^3 + y^3 + e^{2z} = f'(z) \quad \text{--- ④}$$

$$f(z) + e^{2z} = f'(z) \qquad [\text{From ②}]$$

$$f(z) + e^{2z} = \frac{df}{dz}$$

$$\frac{df}{dz} - f(z) = e^{2z} \quad \text{--- ⑤}$$

which is linear D.E.

$$\text{I.f.} = e^{\int -dz} = e^{-z}$$

The general solution of ⑤ is given by

$$f \cdot e^{-z} = \int e^{2z} \cdot e^{-z} dz + c$$

$$f e^{-z} = e^{z} + c$$

$$f(z) = e^{2z} + ce^{z}$$

Putting value of f(z) in ②, we have

$$x^3 + y^3 = e^{2z} + ce^{z}$$

which is the required solution of ①.