# Differential Equations

## UNIT- II

By

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# Method of Change of Independent Variables

Consider the second order LDE.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{---} (1)$$

where P, Q, R are functions of x.

Changing independent variable x into z,

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \quad \text{---} (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$\cancel{= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}}$$

$$= \left(\frac{dz}{dx}\cdot\frac{d^2y}{dz^2}\right)\frac{dz}{dx} + \left[\frac{dy}{dz}\cdot\frac{d}{dx}\left(\frac{dz}{dx}\right)\frac{dx}{dz}\right]\frac{dz}{dx}$$

$$= \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\frac{dy}{dz} \quad \text{---} (3)$$

Using (2) and (3) in (1),

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R \quad \text{---} (4)$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{---} (5)$$

Comparing eq. (4) and (5), we get

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{(dz/dx)^2}$$

(I) $Q_1 =$ Constant, $P_1 = 0$ (most of the time), then find $z$.
Then find the value of $P_1$ and $R_1$.

(1) Solve :- $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$ --- (1)

(2) ~~Solve~~:- On comparing eq. (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ -- (1)

we have $P = \cot x$, $Q = 4\operatorname{cosec}^2 x$, $R = 0$

Now, let after changing the independent variable $x$ to $z$, then eq (1) becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{--(2)}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{(dz/dx)^2}$$

Now $Q_1 = \frac{4\operatorname{cosec}^2 x}{(dz/dx)^2} = 4$ (say)

$$\frac{dz}{dx} = \operatorname{cosec} x$$

$$\int dz = \int \operatorname{cosec} x \, dx$$

$$z = \log|\operatorname{cosec} x - \cot x| \text{ or } \log \tan \frac{x}{2}$$

$$P_1 = \frac{-\operatorname{cosec} x \cdot \cot x + \operatorname{cosec} x \cdot \cot x}{(\operatorname{cosec} x)^2} = 0$$

$$R_1 = \frac{0}{(\operatorname{cosec} x)^2} = 0$$

Putting values of $P_1, Q_1, R_1$ in eq. (2),

$$\frac{d^2y}{dz^2} + 4y = 0 \quad \text{---} (3)$$

A.E. is $m^2 + 4 = 0$

$m = \pm 2i$

C.F. is $y = c_1 \cos 2z + c_2 \sin 2z$

$$y = c_1 \cos\left(2 \log \tan \frac{x}{2}\right) + c_2 \sin\left(2 \log \tan \frac{x}{2}\right)$$

which is the required G.S. of the given D.E.

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(2) $$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 4x^3 \sin x^2 \quad \text{---}$$

Dividing the eq. throughout by $x$, the D.E. becomes

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2 y = 4x^2 \sin x^2 \quad \text{---} (1)$$

Let the differential equation becomes after changing independent variable $x$ into $z$, then (1) becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (2)}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let $Q_1 = \dfrac{-4x^2}{\left(\frac{dz}{dx}\right)^2} = -4$ (say)

$$\left(\frac{dz}{dx}\right)^2 = x^2$$

$$\frac{dz}{dx} = x$$

$$\int dz = \int x\,dx$$

$$z = \frac{x^2}{2} \quad \text{--- (3)}$$

$$P_1 = \frac{1 - \left(\frac{1}{x}\right)x}{x^2} = 0$$

$$R_1 = \frac{4x^2 \sin x^2}{x^2} = 4\sin x^2 = 4\sin 2z \quad \text{[From (3)]}$$

Putting values of $P_1$, $Q_1$ and $R_1$ in eq. (2), we get

$$\frac{d^2y}{dz^2} - 4y = 4\sin 2z$$

$$\frac{1}{(D^2+a^2)^n}\cos ax = \frac{(-1)^n x^n}{(2a)^n n!}\cos\left(ax+\frac{n\pi}{2}\right)$$

A.E. $m^2-4=0$

$m^2=2$

$m=\pm 2$

$$C.F. = c_1e^{2z}+c_2e^{-2z}$$
$$= c_1e^{x^2}+c_2e^{-x^2}$$

$$P.I. = \frac{1}{D^2-4}\,4\sin 2z$$

$$= 4\frac{1}{D^2-4}\boxed{\sin 2z} = 4\frac{1}{(-4-4)}\sin 2z = \frac{-1}{2}\sin 2z$$
$$= \frac{-1}{2}\sin x^2$$

Therefore, G.S. of the given DE. is

$$y = C.F. + P.I.$$
$$y = c_1e^{x^2}+c_2e^{-x^2}-\frac{1}{2}\sin x^2$$

$\int$ To find. Complete solution of LDE. when one integral part of C.F. is known.

Consider the second order L.D.E. of variable coefficients is

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x) \quad \text{--- (1)}$$

Let one part of C.F. of (1) is $y=u$, then $y=u$ satisfy (1) when $R(x)=0$.

$$\frac{d^2u}{dx^2}+\frac{P\,du}{dx}+Qu=0 \quad \text{--- (2)}$$

(3)

Let $y=uv$ be the complete solution of (1).

then it satisfy (1).

Now $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ --- (4)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2} \quad \text{--- (5)}$$

Substituting values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$ from (3), (4) and (5) in (1), we have

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\cdot\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + Pu\frac{dv}{dx} + v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = R \quad \text{--- (6)}$$

From (2) and (6), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + Pu\frac{dv}{dx} = R$$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \quad \text{--- (7)}$$

Let $\frac{dv}{dx} = p$ or $t$

$$\frac{d^2v}{dx^2} = \frac{dt}{dx}$$

Then (7) becomes $\frac{dt}{dx} + \left(P + \frac{2}{\mu}\frac{d\mu}{dx}\right)t = \frac{R}{\mu}$

which is linear differential equation -- (8) of first order

$$t = \frac{dv}{dx} = f(x)$$

On Integrating, $v = \int f(x)dx$

⊙ Inspection method:-

(i) If $1+P+Q=0$, then $y=e^x$ is one part of C.F.

(ii) If $1-P+Q=0$, then $y=e^{-x}$ is one part of C.F.

(iii) If $m^2+Pm+Q=0$, then one part of C.F. $y=e^{mx}$

(iv) If $P+Qx=0$, then $y=x$ is one part " "

(v) If $2+2Px+Qx^2=0$, then $y=x^2$ is " " " ".

(vi) If $m(m-1)+Pmx+Qx^2=0$, then $y=x^m$ is one

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Solve: $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$

The given ~~DE~~ can be written as:

$$\frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{x+2}{x^2}\right)y = xe^x \quad \text{--- (1)}$$

On comparing ① with the standard form of second order LDE,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \text{ we have}$$

$$P = -\left(1+\frac{2}{x}\right), \quad Q = \frac{1}{x} + \frac{2}{x^2}$$

Now $P + Qx = -\left(1+\frac{2}{x}\right) + x\left(\frac{1}{x}+\frac{2}{x^2}\right)$

$$= -1 - \frac{2}{x} + 1 + \frac{2}{x} = 0$$

$\therefore y = x$ is one part of C.F. of ①.

Let $y = xv$ --- ② be the general solution of ①, therefore it satisfy ①.

Now,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{--- ③}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} \quad \text{--- ④}$$

Putting values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in ①, we get

$$x\frac{d^2v}{dx^2} + \frac{2dv}{dx} - \left(1+\frac{2}{x}\right)\left(v + x\frac{dv}{dx}\right) + \left(\frac{1}{x}+\frac{2}{x^2}\right)xv = xe^x$$

$$x\frac{d^2v}{dx^2} + \left(2 - x - \frac{2x}{x}\right)\frac{dv}{dx} - v - \frac{2v}{x} + v + \frac{2v}{x} = xe^x$$

$$x\frac{d^2v}{dx^2} - x\frac{dv}{dx} = xe^x \quad \text{--- ⑤}$$

Dividing ⑤ throughout by $x$,

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$$

[Can use Constant Coefficients method]

Let $\frac{dv}{dx} = p$

$$\frac{d^2v}{dx^2} = \frac{dp}{dx}$$

$$\therefore \frac{dp}{dx} - p = e^x \quad --- ⑥$$

This is LDE of first order.

$$I.F. = e^{\int dx} = e^{-x}$$

The general solution of ⑥ is given by

$$p.e^{-x} = \int e^x . e^{-x} dx + C_1$$

$$p.e^{-x} = x + C_1$$

$$p = xe^x + C_1 e^x$$

$$\frac{dv}{dx} = xe^x + C_1 e^x$$

$$dv = (x+C_1) e^x dx$$

On integration,

$$\int dv = \int (x+C_1) e^x dx$$

$$v = \int xe^x dx + C_1 \int e^x dx$$

$$\int x^3 \sin 3x \, dx = x^3\left(-\frac{\cos 3x}{3}\right) - (3x^2)\left(-\frac{\sin 3x}{3^2}\right) + \left(6x \frac{\cos 3x}{3^3}\right) - \left(6 \cdot \frac{\sin 3x}{3^4}\right)$$

I function is polynomial and Take alternate signs.

Diff Integ

$$v = xe^x - e^x + c_1 e^x + c_2$$

$$v = (x + c_1)e^x - e^x + c_2$$

Putting the value of $v$ in (2), we have

$$y = (x^2 + c_1 x)e^x - xe^x + xc_2$$

which is the required general solution of given DE.

## Removal of first derivative or Change of dependent variable or reduction to Normal form.

Consider the second order L.D.E.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad \text{--- (1)}$$

Let $y = uv$ be the general solution of (1), when $u$ is not one part of C.F.

$$\frac{dy}{dx} = \frac{u\,dv}{dx} + \frac{v\,du}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{u\,d^2v}{dx^2} + \frac{2dv}{dx}\cdot\frac{du}{dx} + \frac{v\,d^2u}{dx^2}$$

Substituting values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1),

$$\frac{u\,d^2v}{dx^2} + \frac{2du}{dx}\cdot\frac{dv}{dx} + \frac{v\,d^2u}{dx^2} + P\left(\frac{u\,dv}{dx} + \frac{v\,du}{dx}\right) + Quv = R$$

$$u\frac{d^2v}{dx^2} + \frac{2du}{dx}\cdot\frac{dv}{dx} + \frac{Pu\,dv}{dx} + \left(\frac{v\,d^2u}{dx^2} + \frac{Pv\,du}{dx} + Quv\right) = R$$

$$u\left\{\frac{d^2v}{dx^2} + \cancel{\frac{2}{u} + \frac{P\,dv}{dx}}\;\frac{dv}{dx}\left(P + \frac{2}{u}\frac{du}{dx}\right)\right\} + v\left\{\frac{d^2u}{dx^2} + \frac{P\,du}{dx} + Qu\right\} = R$$

(2) ---

For removal of first derivative $\frac{dv}{dx}$,

put $P + \frac{2}{u}\frac{du}{dx} = 0$ --- (2)

$$\int \frac{du}{u} = \int \frac{-P}{2} dx$$

$$\log u = \int \frac{-P}{2} dx$$

$$\boxed{u = e^{-\frac{1}{2}\int P dx}} \quad \text{---} (4)$$

Now the eq. (2) becomes;

$$u\frac{d^2v}{dx^2} + v\left\{\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right\} = R$$

$$\frac{d^2v}{dx^2} + \frac{v}{u}\left\{\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right\} = \frac{R}{u} \quad \text{---} (5)$$

Now $\frac{du}{dx} = \frac{-P}{2}u$

$$\frac{d^2u}{dx^2} = \frac{-1}{2}\frac{d}{dx}(Pu)$$

$$= \frac{-1}{2}\left[u\frac{dP}{dx} + P\frac{du}{dx}\right]$$

$$= \frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}$$

Putting values of $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ in (5),

$$\frac{d^2v}{dx^2} + \frac{v}{u}\left\{\frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx} - \frac{P^2u}{2} + Qu\right\} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + v\left\{Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right\} = \frac{R}{u}$$

$\therefore \dfrac{d^2v}{dx^2} + Iv = S$ --- (6)

where $I = Q - \dfrac{P^2}{4} - \dfrac{1}{2}\dfrac{dP}{dx}$

$S = \dfrac{R}{u}$

Here (6) is called normal form of (1).

Now find the value of v and then G.S. of (1) is $y = uv$

(V) Solve :- $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$ --- (1)

On comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$,

$P = -4x, \quad Q = 4x^2 - 1, \quad R = -3e^{x^2}\sin 2x.$

Let $y = uv$ be the G.S. of (1), then its Normal form is where u and v are func of x.

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{--- (2)}$$

where $I = Q - \dfrac{P^2}{4} - \dfrac{1}{2}\dfrac{dP}{dx}$, $S = \dfrac{R}{u}$

$S = \dfrac{R}{u} = \dfrac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$

$I = 4x^2 - 1 - \dfrac{1}{4}(-4x)^2 - \dfrac{1}{2}\dfrac{d(-4x)}{dx}$

$= 4x^2 - 1 - 4x^2 + 2 = 1$

Putting values of S and I in (2),

$$\frac{d^2v}{dx^2} + v = -3\sin 2x \quad \text{--- (3)}$$

which is L.D.E. with constant coefficients.

A.E. $m^2+1=0$

$m^2=-1$

$m=\pm i$

$C.F. = c_1\cos x + c_2 \sin x$

$$P.I. = \frac{1}{D^2+1}(-3\sin 2x) = -3\cdot\frac{1}{D^2+1}\sin 2x$$

$$= \frac{-3}{-4+1}\sin 2x = \sin 2x$$

$v = C.F. + P.I.$

$= c_1\cos x + c_2\sin x + \sin 2x$

$\therefore$ The general solution of (1) is $y = uv$

$$y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x).$$

(2) Solve: $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0$ — (1)

(3) $x^2(\log x)^2\frac{d^2y}{dx^2} - 2x\log x\frac{dy}{dx} + [2+\log x - 2(\log x)^2]y = x^2(\log x)^3$

(2) Comparing the given linear differential equation of second order with the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$P = \frac{1}{x^{1/3}},\quad Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2},\quad R=0$$

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Let $y = uv$ be the general solution of ①
To remove the first derivative,

$$u = e^{-\frac{1}{2}\int P\,dx}$$

$$u = e^{-\frac{1}{2}\int \frac{1}{x^{2/3}}dx} = e^{-\frac{1}{2}\left(\frac{3}{2}x^{2/3}\right)}$$

$$u = e^{-3x^{2/3}}$$

Let The normal form of ① of the given equation be

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{---} ②$$

$$I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} = \frac{1}{4x^{4/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{4}\left(\frac{1}{x^{2/3}}\right)^2$$

$$-\frac{1}{2}\left(\frac{-1}{3x^{4/3}}\right)$$

$$I = \cancel{\frac{1}{4x^{2/3}}} - \frac{6}{x^2} - \cancel{\frac{1}{4x^{2/3}}} \quad \text{---} ③$$

$$S = \frac{R}{u} = \frac{0}{e^{-3x^{2/3}}} = 0 \quad \text{--} ④$$

$\therefore$ Using values of $I$ and $S$ from ③ and ④ in ②, we have

$$\frac{d^2v}{dx^2} + \left(\cancel{\frac{1}{4x^{2/3}}} - \frac{6}{x^2} - \cancel{\frac{1}{4x^{2/3}}}\right)v = 0$$

$$x^2\frac{d^2v}{dx^2} - 6v = 0 \quad \text{---} ⑤$$

⑤ is homogeneous linear differential Equation,
Let $x = e^z \Rightarrow z = \log x$

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$$D \equiv \frac{d}{dz}$$

$$D(D-1)v - 6v = 0$$

$$(D^2 - D - 6)\, v = 0 \quad \text{--- (5)}$$

A.E. is $m^2 - m - 6 = 0 \Rightarrow (m+2)(m-3) = 0$

$m = -2, 3$

C.F. ~~is~~ $v = c_1 e^{-2z} + c_2 e^{3z}$

Therefore the solution of (5) is $\quad$ P.I. $= 0$

$$v = \text{C.F.} + \text{P.I.}$$

$$v = c_1 e^{-2z} + c_2 e^{3z}$$

$$v = c_1 x^{-2} + c_2 x^3$$

The general solution of (1) is given by

$$y = uv$$

$$y = e^{-3/4} x^{2/3} (c_1 x^{-2} + c_2 x^3).$$

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Method of Variation of Parameters

The 2nd order L.D.E. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— (1)

$y = au + bv$ [By putting $R=0$]

be its C.F. where a and b are arbitrary constant

~~Let y =~~

Since u and v are parts of C.F., ~~therefore it satisfy~~ (1) when $R=0$.

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad \text{--- (2)}$$

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \quad \text{--- (3)}$$

Let $y = A(x)u + B(x)v$ —(4) be the G.S. of (1).

Differentiating (4) w.r.t. $x$,

$$\frac{dy}{dx} = u\frac{dA}{dx} + A\frac{du}{dx} + B\frac{dv}{dx} + v\frac{dB}{dx} \quad \text{--- (5)}$$

For finding value of A and B, we choose A and B in such a way that

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \quad \text{--- (6)}$$

Using (6) in (5), we get

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} \quad \text{--- (7)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(A\frac{du}{dx} + B\frac{dv}{dx}\right)$$

$$= \frac{dA}{dx}\frac{du}{dx} + A\frac{d^2u}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} + B\frac{d^2v}{dx^2} \quad \text{--- (8)}$$

Putting values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ from ④, ⑦ and ⑧ in ①, we have

$$\left(A\frac{d^2u}{dx^2} + \frac{du}{dx}\cdot\frac{dA}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\frac{dv}{dx}\right) +$$

$$P\left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + Q(Au + Bv) = R$$

$$A\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + B\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right]$$

$$+ \left(\frac{du}{dx}\cdot\frac{dA}{dx} + \frac{dv}{dx}\cdot\frac{dB}{dx}\right) = R \quad \text{[From ② \& ③]}$$

$$\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} = R$$

$$u_1 A_1 + v_1 B_1 = R \quad \text{--- ⑨}$$

$$uA_1 + vB_1 = 0 \quad \text{--- ⑩} \quad \text{[From ⑥]}$$

$$\boxed{u_1 = \frac{du}{dx},\ v_1 = \frac{dv}{dx} \quad A_1 = \frac{dA}{dx},\ B_1 = \frac{dB}{dx}}$$

① Solve: $\frac{d^2y}{dx^2} + a^2y = \sec ax$ -- ①

On comparing the D.E. ① with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 0,\ Q = a^2,\ R = \sec ax$

A.E. is $m^2 + a^2 = 0$

$$m^2 = -a^2$$

$$m = \pm ai$$

C.F. $= c_1 \cos ax + c_2 \sin ax$

$\quad\quad\quad\quad\downarrow u \quad\quad\quad \downarrow v$

Let $y = A(x)\cos ax + B(x)\sin ax$ --- (1(a)) be the G.S. of (1).

Here, Let $u = \cos ax$ ; $v = \sin ax$.

For finding value of A and B, we have

$$uA_1 + vB_1 = 0$$

$$u_1A_1 + v_1B_1 = R$$

$$A_1 \cos ax + B_1 \sin ax = 0 \quad \text{--- (2)}$$

$$-A_1 a\sin ax + aB_1 \cos ax = \sec ax \quad \text{--- (3)}$$

Multiply (2) by $a\sin ax$ and (3) by $\cos ax$ and adding them

$$aA_1 \sin ax \cos ax + aB_1 \sin^2 ax - aA_1 \sin ax \cos ax + aB_1 \cos^2 ax = 1$$

$$B_1(\sin^2 ax + \cos^2 ax) = \frac{1}{a}$$

Using value of $B_1$ in (2),

$$A_1 = \frac{-1}{a}\tan ax$$

$$\frac{dA}{dx} = \frac{-1}{a}\tan ax$$

$$A = \frac{1}{a^2}\log\cos ax + c_1'$$ where $c_1'$ is arbitrary constant

and $\frac{dB}{dx} = \frac{1}{a} \Rightarrow B = \frac{x}{a} + c_2'$ where $c_2'$ is arbitrary con

Putting values of A and (B) in 1(a), we get

$$y = \left(\frac{1}{a^2}\log\cos ax + c_1'\right)\cos ax + \left(\frac{x}{a} + c_2'\right)\sin ax$$

(2) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$

(3) $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x$

(4) $(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2$

(2) The given D.E. can be written as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x \quad \text{--- (1)}$$

$$P = \frac{1}{x}, \quad Q = \frac{-1}{x^2}, \quad R = e^x$$

Taking RHS = 0, (1) is homogeneous D.E. where $x = e^z$.

$D \equiv \frac{d}{dz}$, we have $[D(D-1) + D - 1]y = 0$

$$(D^2 - 1)y = 0$$

A.E. is $m^2 - 1 = 0$

$$m^2 = 1 \Rightarrow m = \pm 1$$

$$C.F. = c_1 e^z + c_2 e^{-z} = c_1 x + \frac{c_2}{x}$$

$$y = c_1 x + \frac{c_2}{x}$$

Now let the solution of (1) be

$$y = Ax + \frac{B}{x} \quad \text{--- (2)}$$, where A and B are functions of x.

Here let $u = x$, $v = \frac{1}{x}$.

The functions A and B are given by

$$u A_1 + v B_1 = 0 \quad --- (3)$$
$$u_1 A_1 + v_1 B_1 = R \quad --- (4)$$

where $A_1 = \frac{dA}{dx}$, $B_1 = \frac{dB}{dx}$, $u_1 = \frac{du}{dx}$, $v_1 = \frac{dv}{dx}$.

The eq. (3) and 4 becomes,

$$x \frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} = 0 \quad -- (5)$$

$$\frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} = e^x \quad -- (6)$$

Multiply eq. (6) with $x$ and add to (5), we have

$$2x \frac{dA}{dx} = x e^x$$

$$\frac{dA}{dx} = \frac{e^x}{2} \quad -- (7)$$

On integration,

$$\int dA = \int \frac{e^x}{2} dx$$

$$A = \frac{e^x}{2} + k_1 \quad --- (8)$$

Using (7) in (5), $\frac{x \cdot e^x}{2} + \frac{1}{x}\frac{dB}{dx} = 0$

$$\frac{dB}{dx} = -\frac{x^2 e^x}{2}$$

$$\int dB = \int \frac{-1}{2} x^2 e^x dx \Rightarrow B = \frac{-1}{2}\left[x^2 e^x - 2x e^x + 2e^x\right] + k_2 \quad -- (9)$$

Substituting the values of A and B from (8) and (9) in (2), we have

$$y = \left(\frac{e^x}{2} + k_1\right)x - \frac{1}{2}\left[xe^x - 2e^x + \frac{2e^x}{x}\right] + \frac{k_2}{x}$$

$$y = k_1 x + e^x - \frac{e^x}{x} + \frac{k_2}{x}$$

which is the required general solution of the given DE.

(3) Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 1-\cot x$, $Q = -\cot x$, $R = \sin^2 x$

~~Taking RHS of (1) equal to 0, the given DE is homogeneous. Taking $x = e^z$ and $D \equiv \frac{d}{dz}$ the D equation (1) reduces to the following form, $D(D-1)$~~

$$\frac{d^2y}{dx^2} + (1-\cot x)\cdot\frac{dy}{dx} - y\cot x = 0 \quad \text{--- (2)}$$

Now $1 - P + Q = 1 - (1-\cot x) + (-\cot x)$
$= 0.$

$\therefore y = e^{-x}$ is one part of C.F.

Now let $y = ve^{-x}$ --- (3)

Differentiating (3) w.r.t. $x$, we have

$$\frac{dy}{dx} = -ve^{-x} + e^{-x}\frac{dv}{dx} \quad \text{--- (4)}$$

$$\frac{d^2y}{dx^2} = ve^{-x} + \left(-e^{-x}\frac{dv}{dx}\right) - e^{-x}\frac{dv}{dx} + e^{-x}\frac{d^2v}{dx^2} \quad \text{--- (5)}$$

Substituting these values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$

in ②, we have

$$e^{-x}\left[\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right] + (1-\cot x)e^{-x}\left(\frac{dv}{dx} - v\right)$$
$$- ve^{-x}\cot x = 0$$

$$\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} + (v - v + v\cot x)$$
$$- v\cot x = 0$$

$$\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = p$ , $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

$$\frac{dp}{dx} - (1+\cot x)p = 0 \quad \text{---} ⑥$$

which is L.D.E. of first order.

$$I.F. = e^{-\int(1+\cot x)dx} = e^{-x}.e^{\log \sin x}$$
$$= e^{-x}\sin x$$

The solution of ⑥ is given by

$$p.\, e^{-x}\sin x = \int .$$

$$\frac{dp}{dx} = (1+\cot x)p$$

$$\frac{dp}{p} = (1+\cot x)\,dx$$

On integration,

$$\log p = x + \log \sin x + \log c_1$$
$$p = e^{x}\sin x\, c_1$$

$$\frac{dv}{dx} = e^{x} c_1 \sin x$$

Integrating, $v = \frac{1}{2} c_1 e^{x}(\sin x - \cos x) + c_2$ --- (7)

$\therefore$ From (3) and (7), we get

$$y = \frac{1}{2} c_1 (\sin x - \cos x) + c_2 e^{-x}$$

which is C.F. of the given D.E (1).

Let the solution of (1) be

$$y = A(\sin x - \cos x) + Be^{-x} \quad \text{--- (8)}$$

where A and B are functions of $x$ which are given by.

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \quad \text{--- (9)}$$

$$u_1\frac{dA}{dx} + v_1\frac{dB}{dx} = R \quad \text{--- (10)}$$

Here $u_1 = \frac{du}{dx}$, $v_1 = \frac{dv}{dx}$, $u = \sin x - \cos x$
$v = e^{-x}$

Using these values.

$\therefore$ From (9) and (10), we get

$$(\sin x - \cos x)\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \quad \text{--- (11)}$$

$$(\cos x + \sin x)\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \sin^2 x \quad \text{--- (12)}$$

Adding (11) and (12),

$$2\sin x\frac{dA}{dx} = \sin^2 x \Rightarrow \frac{dA}{dx} = \frac{\sin x}{2}$$

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$$\int dA = \frac{1}{2}\int \sin x \, dx$$

$$A = -\frac{1}{2}\cos x + k_1 \quad \text{--- (13)}$$

$$\int dB = \int \left(\frac{e^x \cdot \sin 2x}{4}\right) dx \; (I_1)$$

$$B = -\int \frac{e^x}{4} dx + \int \frac{e^x \cos 2x}{4} dx \; (I_2) + k_2 \quad \text{--- (14)}$$

Using value of $\frac{dA}{dx}$ in (11),

$$(\sin x - \cos x)\left(\frac{\sin x}{2}\right) + e^{-x}\frac{dB}{dx} = 0$$

$$\frac{dB}{dx} = e^x\left[\frac{\cos x \sin x}{2} - \sin^2 x\right]$$

$$\int dB = \int \left\{\frac{e^x}{4}\sin 2x - \frac{e^x}{2}\left(\frac{1-\cos 2x}{2}\right)\right\} dx$$

$$I_1 = \frac{1}{4}\left[\sin 2x \cdot e^x - \int (2\cos 2x \cdot e^x)\, dx\right]$$

$$I_1 = \frac{e^x}{4}\sin 2x - \frac{1}{2}\left[\cos 2x \cdot e^x - \int (-2\sin 2x) e^x \, dx\right]$$

$$I_1 = \frac{e^x}{4}\sin 2x - \frac{e^x}{2}\cos 2x - 4\left\{\frac{1}{4}\int \sin 2x \cdot e^x \, dx\right\}$$

$$I_1 = \frac{e^x}{4}\sin 2x - \frac{e^x}{2}\cos 2x - 4I_1$$

$$I_1 = \frac{e^x}{5}\left[\frac{\sin 2x - 2\cos 2x}{4}\right] \quad \text{----}$$

$$I_2 = \frac{1}{4}\left[\cos 2x \cdot e^x - \int (-2\sin 2x \cdot e^x)\, dx\right]$$

$$= \frac{1}{4}e^x \cdot \cos 2x + \frac{1}{2}\left[\sin 2x \cdot e^x - \int (2\cos 2x \cdot e^x)\, dx\right.$$

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$$I_2 = \frac{e^x}{4}(\cos 2x + 2\sin 2x) - 4I_2$$

$$4I_2 + I_2 = \frac{e^x}{4}(\cos 2x + 2\sin 2x)$$

$$I_2 = \frac{e^x}{20}(\cos 2x + 2\sin 2x)$$

$\therefore$ From (14), we have

$$B = \frac{e^x}{20}[\sin 2x - 2\cos 2x] - \frac{e^x}{4} + \frac{e^x}{20}[\cos 2x + 2\sin 2x]$$

$$B = \frac{e^x}{20}[3\sin 2x - \cos 2x - 5e^x] + k_2 \quad \text{--- (15)}$$

Substituting values of A and B from (13) and (15) in (8), we have

$$y = \left(\frac{-1}{2}\cos x + k_1\right)(\sin x - \cos x) + \left(\frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x - \frac{1}{4}e^x\right) + k_2 e^{-x}$$

$$= \left(\frac{-1}{2}\cos x \sin x - \frac{1}{2}\cos^2 x\right) + k_1(\sin x - \cos x) + \left(\frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x - \frac{1}{4}e^x\right) + k_2 e^{-x}$$

$$= \frac{-1}{10}\sin 2x - \frac{1}{2}\left(\frac{1+\cos 2x}{2}\right) - \frac{1}{20}(\cos 2x) - \frac{1}{4}e^x + k_1(\sin x - \cos x) + k_2 e^{-x}$$

$-\frac{1}{4}$ $-\frac{1}{20}$

$$y = \frac{-1}{10}\sin 2x - \frac{1}{4} - \frac{3}{20}\cos 2x - \frac{1}{4}e^{x} + k_1(\sin x - \cos x) + k_2 e^{-x}$$

which is the req. G.S. of given D.E.

(4) The given D.E. can be written as:

$$\frac{d^2y}{dx^2} + \left(\frac{x}{1-x}\right)\frac{dy}{dx} - \left(\frac{1}{1-x}\right)y = (1-x) \quad \text{--- (1)}$$

On Comparing with standard form of LDE of 2nd order

~~where~~

$$P = \frac{x}{(1-x)}, \quad Q = \frac{-1}{1-x}, \quad R = (1-x)$$

Here $P + Qx = \frac{x}{1-x} - \frac{x}{1-x} = 0$

$\therefore$ $y = x$ is one part of C.F. of (1).

Also $1 + P + Q = 1 + \frac{x}{1-x} - \frac{1}{1-x}$

$$= \frac{1-x+x-1}{1-x} = 0$$

$\Rightarrow$ $y = e^x$ is also a part of C.F.

$\therefore$ The ~~solution~~ C.F. of (1) is $y = c_1 e^x + c_2 x$ where $c_1$ & $c_2$ are arb.

~~Therefore~~ Now Let the solution of (1) be

$y = Ae^x + Bx$ --- (2)

where A and B are functions of x. whose values can be obtained from the following equations:

$e^x A_1 + x B_1 = 0$ --- (3) where $A_1 = \frac{dA}{dx}$

$e^x A_1 + B_1 = (1-x)$ --- (4) $B_1 = \frac{dB}{dx}$

Subtracting ③ from ④, we get

$$(1-x)B_1 = 1-x$$

$$B_1 = \frac{1-x}{1-x} = 1$$

$$\frac{dB}{dx} = 1$$

$$\int dB = \int dx \Rightarrow B = x + c_1' \quad \text{---} ⑤$$

Using value of $B_1$ in eq. ③, we have

$$e^x A_1 + x(1) = 0$$

$$A_1 = -xe^{-x}$$

$$\int dA = \int -xe^{-x} dx$$

$$A = -\left[-xe^{-x} - \int 1(-e^{-x}) dx\right]$$

$$A = xe^{-x} - \int e^{-x} dx$$

$$A = xe^{-x} + e^{-x} + c_2' \quad \text{---} ⑥$$

Substituting values of A and B from ⑤ and ⑥ in ②, we have

$$y = (x+1) + (x+c_1')x + c_2' e^x$$

$$y = c_1' x + c_2' e^x + (x^2 + x + 1)$$

which is the req. G.S. of the given L.D.E. of second order.

$$B_2 = \frac{1}{4}\left[\frac{2(x+1)}{(x+2)} + \frac{1}{2(x+2)^2}\right]e^{x}$$

Method of Operational factors / (Undetermined Coefficients)

① Solve :- $3x^2\dfrac{d^2y}{dx^2} + (2+6x-6x^2)\dfrac{dy}{dx} - 4y = 0$ --- ①

Let $D \equiv \dfrac{d}{dx}$

Then the given D.E. can be written as

$$\left[3x^2D^2 + (2+6x-6x^2)D - 4\right]y = 0$$

$$\left[3x^2D^2 + (2+6x)D - 2(3x^2D+2)\right]y = 0$$

$$\left[D(3x^2D+2) - 2(3x^2D+2)\right]y = 0$$

## Method of Operational factors (Undetermined Coefficients)

(1) Solve :- $3x^2\frac{d^2y}{dx^2} + (2+6x-6x^2)\frac{dy}{dx} - 4y = 0$ --- (1)

Let $D \equiv \frac{d}{dx}$

Then the given D.E. can be written as

$$\left[3x^2D^2 + (2+6x-6x^2)D - 4\right]y = 0$$

$$\left[3x^2D^2 + (2+6x)D - 2(3x^2D+2)\right]y = 0$$

$$\left[D(3x^2D+2) - 2(3x^2D+2)\right]y = 0$$

$(D-2)(3x^2D+2)y=0$ --- ② ✓ $\quad \cancel{(3x^2D+2)(D-2)y=0}$

Let $(3x^2D+2)y = v$ --- ③, then ② becomes

$(D-2)v=0$

$$\frac{dv}{dx}-2v=0 \Rightarrow \int \frac{dv}{v} = 2\int dx$$

$$\log v = 2x + \log c_1$$

$$v = c_1 e^{2x}$$

Substituting values of $v$ in ③,

$$(3x^2D+2)y = c_1e^{2x}$$

$$3x^2\frac{dy}{dx}+2y = c_1e^{2x}$$

$$\frac{dy}{dx}+\frac{2}{3x^2}y = \frac{c_1e^{2x}}{3x^2} \quad \text{--- ④}$$

This is L.D.E. of first order.

$$\text{I.F.} = e^{\int \frac{2}{3x^2}dx} = e^{\frac{-2}{3x}}$$

∴ The solution of ④ is given by

$$y.e^{\frac{-2x}{3}} = \int \frac{e^{\frac{-2x}{3}}.c_1e^{2x}}{3x^2}dx + c_2$$

$$= \frac{c_1}{3}\int e^{\frac{4}{3}x}.x^{-2}dx + c_2$$

$$= \frac{c_1}{3}\left[x^{-2}.\frac{3}{4}e^{\frac{4}{3}x} - \int(-2x^{-3})\frac{3}{4}e^{\frac{4}{3}x}dx\right]$$

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Solve: $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (1+x)e^x$

Let $D \equiv \frac{d}{dx}$, then the given D.E. becomes

$$[(x+2)D^2 - (2x+5)D + 2]y = (1+x)e^x \quad \text{---} (1)$$

$$[xD^2 + 2D^2 - 2xD - 5D + 2]y = (1+x)e^x$$

$$[xD^2 - 2xD + 2D^2 - 4D - D + 2]y = (1+x)e^x$$

$$[xD(D-2) + 2D(D-2) - (D-2)]\cdot y = (1+x)e^x$$

$\cancel{(D-2)(xD+2D-1)y}$

$$(xD+2D-1)(D-2)y = (1+x)e^x$$

$$[(x+2)D-1](D-2)y = (1+x)e^x \quad \text{--} (2)$$

Let $(D-2)y = v$ --(*) then (2) becomes

$$[(x+2)D-1]v = (1+x)e^x$$

$$(x+2)\frac{dv}{dx} - v = (1+x)e^x$$

$$\frac{dv}{dx} - \frac{1}{x+2}v = \frac{(x+1)}{x+2}e^x \quad \text{--- (3)}$$

This L.D.E. of first order.

$$I.F. = e^{\int P dx} = e^{\int \frac{-1}{x+2}dx} = e^{-\log(x+2)} = \frac{1}{x+2}$$

$\therefore$ The solution of (3) is given by

$$v.\frac{1}{x+2} = \int \frac{1}{x+2}.\frac{(x+1)}{x+2}e^x dx + c_1$$

$$\cancel{Let}\ v.\frac{1}{x+2} = \int \frac{(x+1)e^x}{(x+2)^2}dx + c_1$$

$$v.\frac{1}{x+2} = \int \frac{x+2-1}{(x+2)^2}e^x dx + c_1$$

$$v.\frac{1}{x+2} = \int \left(\frac{1}{x+2} - \frac{1}{(x+2)^2}\right)e^x dx + c_1$$

$$v.\frac{1}{x+2} = \frac{e^x}{x+2} + c_1 \qquad \left[\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + c\right.$$

$$v = e^x + c_1(x+2) \quad \text{--- (4)}$$

$\therefore$ Substituting value of $v$ in equation (*), we have

$$(D-2)y = e^x + c_1(x+2)$$

$$\frac{dy}{dx} - 2y = e^x + c_1(x+2) \quad \text{--- (5)}$$

This is L.D.E. of first order.

$$I.F. = e^{\int 2dx} = e^{-2x}$$

The solution of (5) is given by

$$y.e^{-2x} = \int e^{-x}dx + c_1\int (x+2)e^{-2x}dx + c_2$$

$$ye^{-2x} = -e^{-x} + c_1\left[(x+2)\left(\frac{e^{-2x}}{-2}\right) - (1).\left(\frac{e^{-2x}}{4}\right)\right] + c_2$$

$$y.e^{-2x} = -e^{-x} + c_1\left[\frac{-xe^{-2x}}{2} - e^{-2x} - \frac{e^{-2x}}{4}\right]$$

$$= -e^{-x} + c_1\left[\frac{-2xe^{-2x} - 4e^{-2x} - e^{-2x}}{4}\right] + c_2$$

$$y.e^{-2x} = -e^{-x} - \frac{c_1}{4}\left[2x+5\right]e^{-2x} + c_2$$

$$y = -e^{x} - \frac{c_1}{4}(2x+5) + c_2e^{2x}$$

which is the required G.S. of given D.E.

---

## Exact Differential Eq$^n$ of $n^{th}$ order

Let $n^{th}$ order L.D.E. is

$$P_0.\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + P_ny = Q(x) \quad \text{---(1)}$$

where $P_0, P_1, \dots, P_n, Q$ are functions of $x$.

Let first integral form of (1) is.

$$P_0\frac{d^{n-1}y}{dx^{n-1}} + Q_1\frac{d^{n-2}y}{dx^{n-2}} + \dots + Q_{n-1}y = \int Q(x)dx + c$$

where $Q_1, Q_2, \dots, Q_{n-1}$ are functions of $x$. (2)

Now differentiating (2) w.r.t. $x$, we have.

$$P_0\frac{d^ny}{dx^n} + P_0'\frac{d^{n-1}y}{dx^{n-1}} + Q_1\frac{d^{n-1}y}{dx^{n-1}} + Q_1'\frac{d^{n-2}y}{dx^{n-2}}$$

$$+ \dots + Q_{n-1}\frac{dy}{dx} + Q_{n-1}'y = Q(x)$$

$$P_0\frac{d^ny}{dx^n} + (P_0' + Q_1)\frac{d^{n-1}y}{dx^{n-1}} + (Q_1' + Q_2)\frac{d^{n-2}y}{dx^{n-2}}$$

$$+ \dots + (Q_{n-2}' + Q_{n-1})\frac{dy}{dx} + Q_{n-1}'y = Q(x) \quad \text{--- (3)}$$

D-Eq (1) & (3) are same (exact D.E.). Therefore on comparing them,-

$$P_0 = P_0$$

$$P_1 = P_0' + Q_1 \implies Q_1 = P_1 - P_0'$$

$$P_2 = Q_1' + Q_2 \implies Q_2 = P_2 - Q_1'$$
$$= P_2 - P_1' + P_0''$$

$$Q_3 = P_3 - P_2' + P_1'' - P_0'''$$

$$Q_{n-1} = P_{n-1} - P_{n-2}' + P_{n-3}'' - P_{n-4}''' + \dots (-1)^{n-1}P_0^{(n-1)}$$

$$P_n = Q_{n-1}'$$

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$P_n - Q'_{n-1} = 0$

$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \dots + (-1)^n P_0^{\,n} = 0$

$\therefore$ which is exactness condition for $n^{th}$ order D.E. (1).

3/2/20 Solve: $(1+x+x^2)\frac{d^3y}{dx^3} + (3+6x)\frac{d^2y}{dx^2} + \frac{6dy}{dx} = 0$

On comparing given D.E. equation with --(1)

$P_0\frac{d^3y}{dx^3} + P_1\frac{d^2y}{dx^2} + P_2\frac{dy}{dx} + P_3 y = Q(x)$

$P_0 = 1+x+x^2$; $P_1 = 3+6x$, $P_2 = 6$, $P_3 = 0$

Now $P_3 - P'_2 + P''_1 - P'''_0 = 0-0+0-0=0$

$\therefore$ Given D.E. is exact whose first integral form is given by

$$P_0\frac{d^2y}{dx^2} + Q_1\frac{dy}{dx} + Q_2 y = \int Q\,dx + c_1 \quad \text{--(2)}$$

$Q_1 = P_1 - P'_0 = 3+6x-(1+2x)$
$= 2+4x$

$Q_2 = P_2 - P'_1 + P''_0 = 6-6+2 = 2.$

Putting values in (2), we get

$(1+x+x^2)\frac{d^2y}{dx^2} + (2+4x)\frac{dy}{dx} + 2y = \int 0\,dx + c_1$

$(1+x+x^2)\frac{d^2y}{dx^2} + (2+4x)\frac{dy}{dx} + 2y = c_1$ --(3)

Again on comparing ③ with $P_0\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=Q$, we have

$P_0 = 1+x+x^2, \; P_1 = 2+4x, \; P_2 = 2, \; Q = c_1$

Now $P_2 - P_1' + P_0'' = 2 - (4) + (2) = 0$

$\therefore$ D.E ③ is exact whose first integral form is

$$P_0\frac{dy}{dx} + Q_1 y = \int Q\,dx + c_2 \quad \text{--- (4)}$$

where $Q_1 = P_1 - P_0' = 2+4x-(1+2x)$
$= 1+2x$

Putting values in (4), we have

$$(1+x+x^2)\frac{dy}{dx} + (1+2x)y = c_1x + c_2 \quad \text{--- (5)}$$

Again on comparing ⑤ with $P_0\frac{dy}{dx}+P_1y=Q$,

we have $P_0 = 1+x+x^2, \; P_1 = 1+2x,$
$Q = c_1x+c_2$

Now $P_1 - P_0' = 1+2x-(1+2x) = 0$

$\therefore$ D.E. ⑤ is exact and whose first integral form is

$$P_0y = \int Q\,dx + c_3$$

$$(1+x+x^2)y = \int (c_1x+c_2)\,dx + c_3$$

$$(1+x+x^2)y = \frac{c_1 x^2}{2} + c_2 x + c_3$$

which is required G.S. of given differentia equation (1).

Solve :-

(2) $(2x^2+3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x$

(3) $\frac{d^3y}{dx^3} + \cos x \frac{d^2y}{dx^2} - 2\sin x \frac{dy}{dx} - y\cos x = \sin 2x$