Differential Equations

UNIT- III and IV

By

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Partial D.E.

D.E. → An eqⁿ which represents relationship b/w dependent variable & its derivative w.r.t. independent variable

O.D.E. P.D.E.

① **Quasi-Linear P.D.E. of first order (Lagrange's eqⁿ)**

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

p & q has power 1 and no product of pq

A P.D.E. is said to be quasi-linear P.D.E. if it is linear in p and q and z may be non-linear.

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

Eg. $xyzp + xyz^2 q = x^2y^2$ (✓)

$p + q = 1$ (✓)

$pq + q = xyz$ (×)

② **Semi-linear or almost linear P.D.E.**

A P.D.E. of the form $P(x,y)\,p + Q(x,y)\,q = R(x,y,z)$

Linear P.D.E. of first order → If the P.D.E. has powers of z, p and q equal to 1 and there is no product of them where z is dependent variable, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)\,z + R_1(x,y)$$

Non-linear PDE → A P.D.E. is said to be non-linear P.D.E. if it is not linear i.e. p & q have power > 1 or product of p & q exists or both.

$pq = z$ (✓) $p^2 + q = z$ (✓)

$p + q = z^2$ (X)

Formation of P.D.E. by eliminating arbitrary constants:-

Let $\phi(x,y,z,a,b) = 0$ --- (1) be a relation b/w x, y, z where a & b are arbitrary constants.

In order to eliminate 'a' and 'b', we require 2 independent eqn which can be obtained by differentiating (1) partially wrt x and y.

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \quad \text{--- (3)}$$

Now eliminate a & b from (1), (2) and (3).

The resultant eqn is req. P.D.E.

Note ① In eq. ①, the no. of arbitrary constants to be eliminated is just equal to no. of independent variables. And thus an eqⁿ of first order.

② If No. of arbitrary constants > No. of indep. variables the eqⁿ of second order or higher order will arise.

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pr Find the P.D.E. by the elimination of Arbitrary constants 'a' and 'b' from the eqⁿ.

$$z = ax + a^2y^2 + b \quad \text{--- ①}$$

Differentiating partially ① w.r.t x, we have

$$\frac{\partial z}{\partial x} = a \quad \text{--- ②}$$

Again diff. ① partially w.r.t. y, we have

$$\frac{\partial z}{\partial y} = 2a^2y \quad \text{--- ③}$$

Using ② in ③, we have

$$\frac{\partial z}{\partial y} = 2\left(\frac{\partial z}{\partial x}\right)^2 y.$$

$$q = 2p^2y$$ which is req. P.D.E.

pr 2 Eliminate arbitrary constants from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and verify whether the obtained P.D.E. is linear.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{---(1)}$$

Differentiating (1) partially w.r.t. x & y, we have

$$\frac{2x}{a^2}+\frac{1}{c^2}(2z)\frac{\partial z}{\partial x}=0 \quad \text{---(2)}$$

$$\frac{2y}{b^2}+\frac{1}{c^2}(2z)\frac{\partial z}{\partial x}=0 \quad \text{---(3)}$$

Again differentiating partially (2) and (3) w.r.t. x and y ~~w.r.t.~~ respectively,

$$\frac{2}{a^2}+\frac{1}{c^2}\left[2\left(\frac{\partial z}{\partial x}\right)^2+2z\,\frac{\partial^2 z}{\partial x^2}\right]=0 \quad \text{--(4)}$$

$$\frac{2}{b^2}+\frac{1}{c^2}\left[2\left(\frac{\partial z}{\partial y}\right)^2+2z\,\frac{\partial^2 z}{\partial y^2}\right]=0 \quad \text{---(5)}$$

Eliminating $\frac{1}{c^2}$ between (2) and (4), we have

$$\frac{2x}{a^2}+z\frac{\partial z}{\partial x}\left(\frac{-2/a^2}{\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}}\right)=0$$

$$x\left(\frac{\partial z}{\partial x}\right)^2+xz\frac{\partial^2 z}{\partial x^2}-z\frac{dz}{dx}=0 \qquad xp^2+xzr-zp=0$$

which is the req. P.D.E. of second order.

This is non-linear.

4. Formation of P.D.E. by eliminating arbitrary func^n

Let $u \equiv u(x,y,z)$

$v \equiv v(x,y,z)$

give a relation

$$\phi(u,v) = 0 \quad \text{--- (1)}$$ where ϕ is an arbitrary function.

Differentiating partially (1) w.r.t. x and y, we have

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0 \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0 \quad \text{--- (3)}$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (2) and (3), then the resultant relation is req. PDE.

Ex 3 – Eliminate the function f from $f(x+y+z,\ x^2+y^2+z^2) = 0$. --- (1)

Let $u = x+y~~+z~~$

$v = x^2+y^2+z^2$

$\therefore$ (1) becomes $f(u,v) = 0$ --- (2)

Differentiat. partially (2) w.r.t. x, ~~and y~~ we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

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$$\Rightarrow \frac{\partial f}{\partial u}\left(1 + 1.\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(2x + 2z.\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+p) + \frac{\partial f}{\partial v}(2x+2zp) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+p) = -2\frac{\partial f}{\partial v}(x+zp) \quad \text{--- (3)}$$

where $p = \frac{\partial z}{\partial x}$

P. ~~Diff~~ Differentiating w.r.t y of (2), we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}\left(1 + 1.\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial f}{\partial u}(1+q) = -2\frac{\partial f}{\partial v}(y+zq) \quad \text{--- (4)}$$

where $q = \frac{\partial z}{\partial y}$

Dividing (3) by (4),

$$\frac{(1+p)}{(1+q)} = \frac{x+zp}{y+zq}$$

$$y + zq + yp + zpq = x + zp + xq + zpq$$

$$(y+z)p - (x+z)q = (x-y)$$

N-4 Eliminate the arbitrary func^n from the eq^n:-

$$z = f(x+iy) + F(x-iy) \quad \text{---} (1)$$

Differentiating partially (1) w.r.t x,

$$\frac{\partial z}{\partial x} = \frac{d f(x+iy)}{d(x+iy)} \frac{\partial}{\partial x}(x+iy) + \frac{d F(x-iy)}{d(x-iy)} \frac{\partial}{\partial x}(x-iy)$$

$$\frac{\partial z}{\partial x} = f'(x+iy).(1) + F'(x-iy).(1)$$

Similarly, differentiating partially (1) w.r.t. y, we have

$$\frac{\partial z}{\partial y} = f'(x+iy) \frac{\partial}{\partial y}(x+iy) + F'(x-iy) \frac{\partial}{\partial y}(x-iy)$$

$$\frac{\partial z}{\partial y} = f'(x+iy)\, i - iF'(x-iy) \quad \text{---} (3)$$

Again differentiating partially (2) & (3) w.r.t. x and y respectively.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+iy) + F''(x-iy) \quad \text{--} (4)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+iy)(i)^2 + (-i)^2 F''(x-iy) \quad \text{---}$$

$$\frac{\partial^2 z}{\partial y^2} = -\left[f''(x+iy) + F''(x-iy)\right] \quad \text{--} (5)$$

using (4) in (5),

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$ which is req. P.D.E.

Q.1 $Z = axe^y + \frac{1}{2}a^2e^{2y} + b$

$$\frac{\partial z}{\partial y} = axe^y + \frac{1}{2}a^2(2)e^{2y} \cdot = xae^y + a^2e^{2y} \quad \text{--- ①}$$

$$\frac{\partial z}{\partial x} = ae^y \quad \text{--- ②}$$

Using value of 'a' from ② in ①,

$$\frac{\partial z}{\partial y} = \left(e^{-y}\frac{\partial z}{\partial x}\right)xe^y + e^{-2y}\left(\frac{\partial z}{\partial x}\right)^2 e^{2y}$$

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$$

$q = px + p^2$ which is req. P.D.E.

Q.2

$lx + my + nz = \phi(x^2+y^2+z^2)$

⇒ Some fundamental Definitions:-

Complete Integral - Let general form of P.D.E. is $f(x,y,z,p,q) = 0$ --- ① which can be obtained by eliminating arbitrary a and b from an equation

$\phi(x,y,z,a,b) = 0$ --- ②

where a, b are arbitrary constants

Then ② is called C.I. of ①.

∵ ① is Ist order P.D.E. ⇒ no. of independent variables equal to no. of arbitrary constants, then the solution is said to be C.I.

Particular Integral → A solution of PDE is said to be particular integral if it can be obtained by putting particular values of arbitrary constants in C.I. i.e. P.I. does not contains any arbitrary constants.

General Integral → Let the P.D.E. $f(x, y, z, p, q) = 0$ --- ①

which can be obtained from $\phi(u, v) = 0$ by eliminating arbitrary function ϕ ---②

where u and v are funcn of x, y, z and ϕ is arbitrary funcn.

Then ② is said to be General Integral of P.D.E. ① i.e.

General Integral is the relationship between depen. variable, indepen. variable and arbitrary functions.

~~Particular~~ Singular Integral → A solution of P.D.E. is said to be singular integral if it cannot be obtained by putting particular values of arbitrary constants in its complete integral.

Let $\phi(x, y, z, a, b) = 0$ --- ① be a complete integral of the P.D.E. $f(x, y, z, p, q) = 0$ --- ②

Lagrange's linear $Eq^n \rightarrow Pp + Qq = R$

P, Q, R are func^n of x, y, z.

For singular integral, eliminate from 'a' and 'b' from

$$\phi = 0, \quad \frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0$$ and satisfy the given PDE.

$x+y+a+b=0$ is not S.I.

Lagrange's method (Only linear P.D.E.)

Let quasi-linear PDE of I^{st} order of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z).$$

i.e. $Pp + Qq = R$, then

Lagrange's A.E. (auxiliary eq^n) or characteristic eq^n is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Then general integral $\phi(c_1, c_2) = 0$ where $u = c_1$ and $v = c_2$ are independent sol^n of ① and ϕ is arbitrary func^n.

1. Solve :- $(y+z)p + (z+x)q = x+y$ ...①

On comparing ① with $Pp + Qq = R$, we have

$P = y+z, \quad Q = z+x, \quad R = x+y$

Lagrange's A.E. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} \quad \text{--- (2)}$$

Taking last 2 ratios of (2) fractions

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\int \frac{dx-dy}{-(x-y)} = \int \frac{dy-dz}{-(y-z)}$$

Let $x-y=t$

$dx-dy=dt$

and $y-z=t_2$

$dy-dz=dt_2$

$$\int \frac{dt}{t} = \int \frac{dt_2}{t_2}$$

$$\log t = \log t_2 + \log c_1$$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1 \Rightarrow c_1 = \frac{x-y}{y-z} \quad \text{--- (3)}$$

where c_1 is arbitrary constant

Again by (2),

$$\int \frac{dx-dy}{y-x} = \int \frac{dx+dy+dz}{2(x+y+z)}$$

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) - \log c_2$$

$$c_2 = (x-y)\sqrt{x+y+z} \quad \text{--- (4)}$$

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$\therefore$ General Integral of given PDE is given by
$\phi(c_1, c_2) = 0$

$$\phi\left(\frac{x-y}{y-z}, (x-y)\sqrt{x+y+z}\right) = 0 \text{ is the req. G.S}$$

(2) Solve: $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$ --- (1)

The given P.D.E. can be written as

$$(y^2+z^2-x^2)p - 2xyq = -2xz \quad \text{— (2)}$$

On comparing (1) with $Pp + Qq = R$,

$$P = y^2+z^2-x^2, \quad Q = -2xy, \quad R = -2xz$$

for (2)

Now, Lagrange's A.E. is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \text{--- (2)}$$

Taking last two ratios in (2), we have

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$$\int\frac{dy}{y} = \int\frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\frac{y}{z} = c_1 \quad \text{--- (3)}$$

Again by (2),

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} = \frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)} \quad \text{---} (4)$$

Taking second and fourth ratios in (4),

$$\frac{dy}{-2xy} = \frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$$

$$\int \frac{dy}{y} = \int \frac{2(xdx+ydy+zdz)}{(x^2+y^2+z^2)}$$

$$\log y = \log(x^2+y^2+z^2) + \log c_2$$

$$\log\left(\frac{y}{(x^2+y^2+z^2)}\right) = \log c_2$$

$$y(x^2+y^2+z^2) = c_2 \quad \text{---} (5)$$

$\therefore$ General integral of given P.D.E. is given by

$$\phi(c_1, c_2) = 0$$

where c_1 and c_2 are given by (3) and (5).

3) Solve:- $p\cos(x+y) + q\sin(x+y) = z$ -- (1)

Lagrange's Auxiliary equation

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \quad \text{--- (2)}$$

By taking last two ratios in (2),

$$dx-dy = \frac{[\cos(x+y)-\sin(x+y)](dx+dy)}{\cos(x+y)+\sin(x+y)}$$

Integrating both sides,

$$x-y = \log|\cos(x+y)+\sin(x+y)| + \log c_1$$

$$e^{x-y} = c_1[\cos(x+y)+\sin(x+y)]$$

$$c' = \frac{1}{c_1} = e^{y-x}[\cos(x+y)+\sin(x+y)] \quad \text{--- (3)}$$

Taking third and fourth ratios in (2).

$$\sqrt{2}\int\frac{dz}{z} = \int\frac{dx+dy}{\frac{1}{\sqrt{2}}\cos(x+y)+\frac{1}{2}\sin(x+y)}$$

$$\sqrt{2}\int\frac{dz}{z} = \int\frac{dx+dy}{\sin\frac{\pi}{4}\cos(x+y)+\cos\frac{\pi}{4}\sin(x+y)}$$

$$\sqrt{2}\log z = \int\frac{dx+dy}{\sin(\frac{\pi}{4}+x+y)} \quad \text{--- (4)}$$

Let $\frac{\pi}{4}+x+y = t$

$dt = dx+dy$

$\therefore$ By (4) $\sqrt{2}\log z = \int\frac{dt}{\sin t} \implies \sqrt{2}\log z = \log\left(\tan\frac{t}{2}\right) + \log c_2$

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Integral surface passing through a given curve

$Pp + Qq = R$ --- (1)

Let

$u = c_1$ } u & v func^n of x, y, z

$v = c_2$ } are two solutions of A.E. of (1).

(1) Find the integral surface of PDE :— --- (1)

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which passes through the straight line $x+y=0$, $z=1$.

A.E. of this eq^n is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{--- (2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiples

$$\frac{dx}{x(y^2+z)} =$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = 0 \quad y^2+z \; -x^2-z \; x^2-y^2$$

$\log xyz = \log c_2$

$xyz = c_2$ --- (3)

Taking $\cancel{x}$, y, and $\cancel{1}$ as multipliers

$$\frac{x\,dx + y\,dx - dz}{0} = \frac{dz}{z(x^2-y^2)}$$

$x^2 + y^2 - 2z = c_1$ --- (4)

The ~~Given straight~~ eq of st. line, $x+y=0$, $z=1$

Let $y = t$ $\Rightarrow x = -y$

$$t^2 + t^2 - 2 = c_1 \quad \text{---} (3)$$

$$-t^2 = c_2 \quad \text{---} (4)$$

From (3) & (4),

$$-c_2 - c_2 - 2 = c_1$$

$$-2c_2 - 2 = c_1$$

$$c_1 + 2c_2 + 2 = 0$$

$$x^2 + y^2 - z^2 + 2xyz + 2 = 0$$

2. Find I. Surf. of linear P.D.E. $xp + yq = z$ which contains the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 2$.

The given eq^n is $xp + yq = z$

A.E. is $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$ --- (1)

Taking 1, 1, 1 as multipliers, we get

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{dx + dy + dz}{x + y + z} \quad \text{---} (2)$$

Taking last two ratios in (2)

$$\frac{dz}{z} = \frac{dx + dy + dz}{x + y + z}$$

On integration

$$\log z = \log(x + y + z) + \log c_1$$

$$c_1 = \frac{z}{x + y + z} \quad \text{---} (3)$$

Using $x + y + z = 2$ in (3), $z = 2c_1$

Now we know that

Taking first two ratios,

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating on both sides,

$$\log x = \log y + \log c_1$$

$$\log \frac{x}{y} = \log c_1 \Rightarrow c_1 = \frac{x}{y} \quad \text{--- (2)}$$

Taking last two ratios,

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating both sides,

$$\log y = \log z + \log c_2$$

$$c_2 = \frac{y}{z} \quad \text{--- (3)}$$

The given curve is $x^2+y^2+z^2=4$, $x+y+z=2$

Using $x = c_1 y$ and $z = \frac{y}{c_2}$ from (2) and (3) in $x^2+y^2+z^2=4$ and $x+y+z=2$

we have $c_1^2 y^2 + y^2 + \frac{y^2}{c_2^2} = 4$ --- (4)

$$c_1 y + y + \frac{y}{c_2} = 2 \Rightarrow y\left(c_1 + 1 + \frac{1}{c_2}\right) = 2 \quad \text{--- (5)}$$

Squaring (5) and dividing by (4),

$$\frac{y^2\left(c_1+1+\frac{1}{c_2}\right)^2}{y^2\left(c_1^2+1+\frac{1}{c_2^2}\right)} = \frac{4}{4} \Rightarrow 2c_1 + \frac{2}{c_2} + \frac{2c_1}{c_2} = 0$$

$xy + yz + zx = 0$

Charpit's Method

Let non-linear P.D.E. of 1st order be

$f(x,y,z,p,q) = 0$ ---- (1)

Then, Charpit's A.E. or characteristic eqn is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

OR

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

where $f_x = \frac{\partial f}{\partial x}$, $f_p = \frac{\partial f}{\partial p}$, $f_y = \frac{\partial f}{\partial y}$

$f_q = \frac{\partial f}{\partial q}$, $f_z = \frac{\partial f}{\partial z}$

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(p & q in form of x, y, z)

(2) Solve: $p^2+q^2-2px-2qy+2xy=0$

Let $f \equiv p^2+q^2-2px-2qy+2xy=0$ --- (1)

$f_x = -2p+2y$ $\qquad f_p = 2p-2x$

$f_y = -2q+2x$ $\qquad f_q = 2q-2y$

$f_z = 0$

Charpit's A.E. is $\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p+qf_q}$

$= \dfrac{dp}{-(f_x+pf_z)} = \dfrac{dq}{-(f_y+qf_z)}$

$\dfrac{dx}{p-x} = \dfrac{dy}{q-y} = \dfrac{dz}{p^2+q^2-xp-yq} = \dfrac{dp}{p-y} = \dfrac{dq}{q-x}$

Now by taking $\dfrac{dx}{p-x} = \dfrac{dy}{q-y} = \dfrac{dp}{p-y} = \dfrac{dq}{q-x}$

$= \dfrac{dx+dy}{p+q-(x+y)} = \dfrac{dp+dq}{p+q-(x+y)}$ --- (2)

Now, by taking last two relations in (2),

$dx+dy = dp+dq$

Integrating on both sides, we have

$p+q = x+y+a$, where a is arbitrary constant.

$(p-x)+(q-y) = a$ --- (3)

(1) can be written as

$$(p-x)^2+(q-y)^2-x^2-y^2+2xy=0$$

$$(p-x)^2+(q-y)^2=(x-y)^2 \quad \text{--- (4)}$$

Let $p-x=P$, $q-y=Q$, we have

from (3) and (4),

$$P+Q=a \quad \text{--- (5)}$$

$$P^2+Q^2=(x-y)^2 \quad \text{--- (6)}$$

Now $(P-Q)^2 = P^2+Q^2-2PQ$

$$= P^2+Q^2-\left[(P+Q)^2-(P^2+Q^2)\right]$$

$$(P-Q)^2=(x-y)^2-\left[a^2-(x-y)^2\right]$$

$$P-Q=\left[2(x-y)^2-a^2\right]^{1/2} \quad \text{--- (7)}$$

Adding (5) and (7), we have

$$P=\frac{a}{2}+\frac{1}{2}\left[2(x-y)^2-a^2\right]^{1/2} \quad \text{--- (8)}$$

Subtracting (5) and (7),

$$Q=\frac{a}{2}-\frac{1}{2}\left[2(x-y)^2-a^2\right]^{1/2} \quad \text{--- (9)}$$

From (8),

$$p=x+\frac{a}{2}+\frac{1}{2}\left[2(x-y)^2-a^2\right]^{1/2}$$

From (9),

$$q=y+\frac{a}{2}-\frac{1}{2}\left[2(x-y)^2-a^2\right]^{1/2}.$$

$\because$ We know that $dz = p\,dx + q\,dy$

$$dz = \left\{x + \frac{a}{2} + \frac{1}{2}\left[2(x-y)^2 - a^2\right]^{1/2}\right\}dx + \left\{y + \frac{a}{2} - \frac{1}{2}\left[2(x-y)^2 - a^2\right]^{1/2}\right\}dy$$

$$dz = x\,dx + y\,dy + \frac{a}{2}(dx+dy) + \frac{1}{2}\left[2(x-y)^2 - a^2\right]^{1/2}(dx-dy)$$

Integrating on both sides,

$$z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2}(x+y) + \frac{1}{\sqrt{2}}\left[\frac{x-y}{2}\sqrt{(x-y)^2 - a^2} - \frac{a^2}{2(2)}\log\left|x-y+\sqrt{(x-y)^2+\left(\frac{a}{\sqrt{2}}\right)^2}\right|\right]$$

$$2z = x^2 + y^2 + ax + ay + \frac{1}{\sqrt{2}}\left[(x-y)\sqrt{(x-y)^2 - a^2} - \frac{a^2}{2}\log\left|x-y+\sqrt{(x-y)^2+\left(\frac{a}{\sqrt{2}}\right)^2}\right|\right]$$

③ Solve :-

Let $f \equiv 2(pq + py + qx) + (x^2+y^2) = 0$ --- ①

$$\frac{dp}{-(q+x)} = \frac{dq}{-(p+y)} = \frac{dx}{2(q+y)} = \frac{dy}{2(p+x)}$$

$$\frac{dp + dq + dx + dy}{x+y} = 0$$ --- ②

Integrating both sides

$p + q + x + y = a$ where a is arbitrary constant

From (1)

$$2pq + 2py + 2qx + x^2 + y^2 = 0$$

$$2pq + (y+p)^2 - p^2 + (x+q)^2 - q^2 = 0$$

$$(y+p)^2 + (x+q)^2 - (p-q)^2 = 0$$

$$(y+p)^2 + (x+q)^2 = (p-q)^2$$

Let $y+p = P$, $x+q = Q$,

$\Rightarrow$ $P^2 + Q^2 = (p-q)^2$ --- (3)

and eq. (2) becomes $P+Q = a$ --- (4)

Now $(P-Q)^2 = P^2 + Q^2 - 2PQ$

$$= P^2 + Q^2 - 2\left[(P+Q)^2 - (P^2+Q^2)\right]$$

$$= (p-q)^2 - 2\left[a^2 - (p-q)^2\right]$$

$$P - Q = \left\{(p-q)^2 - 2\left[a^2 - (p-q)^2\right]\right\}^{1/2} \text{ --- (5)}$$

Adding (4) and (5), we have

$$P = \frac{a}{2} + \frac{1}{2}\left[(p-q)^2 - 2\left\{a^2 - (p-q)^2\right\}\right]^{1/2} \text{ --- (6)}$$

Subtracting (4) and (5), we get

$$Q = \frac{a}{2} - \frac{1}{2}\left[(p-q)^2 - 2\left\{a^2 - (p-q)^2\right\}\right]^{1/2} \text{ --- (7)}$$

$p = -y + \frac{a}{2} +$ " " " [From 6]

$q = -x + \frac{a}{2} -$ - - - - [(7)]

We know that $dz = p\,dx + q\,dy$

$$dz = \left\{ -y + \frac{a}{2} + \frac{1}{2}\left[(p-q)^2 - 2\{a^2-(p-q)^2\}\right]^{1/2} \right\} dx + \left\{ -x + \frac{a}{2} - \frac{1}{2}\left[(p-q)^2 - 2\{a^2-(p-q)^2\}\right]^{1/2} \right\} dy$$

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Some Standard form of Charpit's method:-

(I) Eq^n of the form $f(p,q) = 0$ --- (1)

$\because$ f is function of p and q,

$f_x = 0,\ f_y = 0,\ f_z = 0$

Charpit's Auxiliary Equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$$\therefore \frac{dp}{0} = \frac{dx}{f_p} \Rightarrow dp = 0 \Rightarrow p = a \quad \text{...(2)}$$

Similarly $dq = 0 \Rightarrow q = b$ --- (3)

where a, b are arbitrary constants

$\because$ We know that $dz = p\,dx + q\,dy$

$dz = a\,dx + b\,dy$ [By (2) & (3)]

Integrating on both sides,

$$z = ax + by + c \quad \text{--- (4)}$$

Again putting the values of p and q in (1), we have

$$f(a,b)=0$$

$$\Rightarrow b=\phi(a)$$

(4) becomes $Z = ax + \phi(a)y + c$
which is required C.I. of given PDE.

⊙ Solve:- $p^2-q^2=1$ [OR find Singular Solⁿ]

Let $f \equiv p^2-q^2-1=0$ ---(1) which is funcⁿ of p and q.

Let complete integral of (1) is
$Z = ax+by+c$, where a, b, c are arbitrary constants. ---(2)

Put $p=a$ and $q=b$ in (1), we have

$$\therefore a^2-b^2=1$$

$$a = \pm\sqrt{1+b^2}$$ [Interconvertible]

Putting value in (2), we have
(+ve)

$$Z = \sqrt{1+b^2}\,x + by + c$$

which is req. complete integral of (1),

$\Rightarrow$ For Singular Solⁿ. $\frac{\partial Z}{\partial b}=0 \Rightarrow \frac{b}{\sqrt{1+b^2}}x + y = 0$

$\frac{\partial Z}{\partial c}=0 \Rightarrow 1 = 0$ which is absurd condition.

Singular Solution $\leftarrow$ $\therefore$ S.S. does not exist.

Q. Find the C.I. of $(y-x)(qy-px)=(p-q)^2$ --- (1)

$qy^2-qxy-pxy+px^2=(p-q)^2$ ---

Let $x+y=X$, $xy=Y$.

Now $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x}$

$p=\frac{\partial z}{\partial X}\cdot 1+\frac{\partial z}{\partial Y}\cdot y$ --- (2)

and

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$

$=\frac{\partial z}{\partial X}\cdot 1+\frac{\partial z}{\partial Y}\cdot x$ --- (3)

Now let $\frac{\partial z}{\partial X}=P$, $\frac{\partial z}{\partial Y}=Q$

$p=P+Qy$ --- (4)

$q=P+Qx$ --- (5)

Putting values of p and q in (1), we have

$\cancel{(y-x)}\left[(P+Qx)y-(P+Qy)x\right]=(y-x)^{\cancel{2}}Q^2$

$Py-Px=(y-x)Q^2$

$P=Q^2$

$\Rightarrow \underset{(say)}{f}\equiv P-Q^2=0$ --- (6)

Let complete integral of (6) be $z=aX+bY+c$ --- (7)

Put $P = a$ and $Q = b$ in (6), we have

$$a - b^2 = 0$$

$$a = b^2$$

$$Z = b^2 X + bY + c$$

where a, b, c are arbitrary consts.

$\because x + y = X$

$xy = Y$

$$Z = b^2(x+y) + bxy + c$$

(II) Equation of the form $f(z, p, q) = 0$

Given PDE $f(z, p, q) = 0$ --- (1)

Let tentative solⁿ of (1) be

$$Z = f(u) = f(x + ay) \quad \text{---}(2)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial f(x+ay)}{\partial x}$$

f depends only on u

$$p = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{df}{du} = \frac{dz}{du} \quad \text{--}(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} \cdot a$$

$$\therefore q = a\frac{df}{du} = a\frac{dz}{du} \quad \text{--}(4)$$

$\because$ ~~We know that~~ $dz = p\,dx + q\,dy$

Substituting values of p and q from (3) and (4) in (1), we have

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \quad \cdots (5)$$

which is ordinary DE in $\frac{dz}{du}$.

⊙ Find the CI of $p^2z^2 + q^2 = 1$.

Let $f \equiv p^2z^2 + q^2 - 1 = 0 \quad \cdots (1)$

The eq (1) is of the form $f(z, p, q) = 0$.

Let tentative solution of (1) be

$$z = f(u) = f(x + ay) \quad \cdots (2) \quad \text{where } a \text{ is arbitrary constant}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \quad \cdots (3)$$

$$q = \frac{\partial z}{\partial y} = a\frac{dz}{du} \quad \cdots (4)$$

Putting values of p and q in (1), we have

$$\left(\frac{dz}{du}\right)^2 z^2 + \left(a\frac{dz}{du}\right)^2 = 1$$

$$\left(\frac{dz}{du}\right)^2 \left[z^2 + a^2\right] = 1$$

Quadratic formula & Factorization

$$\frac{dz}{du} = \pm \frac{1}{\sqrt{z^2 + a^2}}$$

Taking +ve sign,

$\sqrt{z^2 + a^2}\, dz = du$ Integrating on both sides

$$\frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2}\log|z+\sqrt{z^2+a^2}| = \frac{u+b}{2}$$

$$\frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2}\log|z+\sqrt{z^2+a^2}| = x+ay+b$$

which is CI of (1) where a and b are arbitrary constants.

- Solve: $pz = (1+q^2)$

(III) Equation of the form $f_1(x,p) = f_2(y,q)$

~~Let the trial solution be~~

$$f_1(x,p) = f_2(y,q) = a \text{ (say) arbitrary constant 'a'}$$

Now, Equating each side to a, we have

$$f_1(x,p) = a \Rightarrow p = \phi_1(x,a)$$

and $f_2(y,q) = a \Rightarrow q = \phi_2(y,a)$

Since we know that $dz = p\,dx + q\,dy$

$$dz = \phi_1(x,a)\,dx + \phi_2(y,a)\,dy$$

On integrating,

$$z = \int \phi_1(x,a)\,dx + \int \phi_2(y,a)\,dy + c$$

- Solve: $p^2+q^2-2px-2qy+1=0$ --- (1)

Let $p^2-2px+1 = 2qy-q^2 = a$ (say)

Taking $p^2-2px+1 = a$

$p^2 - 2px + 1 - a = 0$

By applying quadratic formula,

$$p = \frac{2x \pm \sqrt{4x^2 - 4(1-a)}}{2} = x \pm \sqrt{x^2 - (1-a)}$$

Now taking

$2qy - q^2 = a$ --- (2)

$$q^2 - 2qy + a = 0$$

$$q = \frac{2y \pm \sqrt{4y^2 - 4a}}{2(1)} = y \pm \sqrt{y^2 - a} \quad \text{--- (3)}$$

∵ We know that

$$dz = p\,dx + q\,dy \quad \text{--- (4)}$$

Substituting values of p and q from (2) and (3) in (4),

$$dz = \left\{x \pm \sqrt{x^2 - (1-a)}\right\} dx + \left\{y \pm \sqrt{y^2 - a}\right\} dy$$

On integrating, we have

$$z = \int \left(x \pm \sqrt{x^2 - (1-a)}\right) dx + \int \left(y \pm \sqrt{y^2 - a}\right) dy$$

Taking +ve sign,

$$z = \frac{x^2}{2} \pm \left[\frac{x}{2}\sqrt{x^2 + a - 1} - \frac{(1-a)}{2}\log\left(x + \sqrt{x^2 - (1-a)}\right)\right]$$

$$+ \frac{y^2}{2} \pm \left[\frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2}\log\left|y + \sqrt{y^2 - a}\right|\right] + b$$

⊙ Solve: $z^2(p^2+q^2) = x^2+y^2$

$$z^2p^2 + z^2q^2 = x^2+y^2 \quad \text{--- (1)}$$

Let $z^2p^2 - x^2 = y^2 - z^2q^2 = a$ (say)

Taking $z^2p^2 - x^2 = a$

$$p^2 = \frac{x^2+a}{z^2} \Rightarrow p = \frac{1}{z}\sqrt{x^2+a} \quad \text{--- (2)}$$

Taking $y^2 - z^2q^2 = a$

$$q^2 = \frac{y^2-a}{z^2} \Rightarrow q = \frac{1}{z}\sqrt{y^2-a} \quad \text{--- (3)}$$

∵ We know that

$$dz = p\,dx + q\,dy \quad \text{--- (4)}$$

Substituting values of p and q from (2) and (3) in (4), we get

$$dz = \left(\frac{1}{z}\sqrt{x^2+a}\right)dx + \left(\frac{1}{z}\sqrt{y^2-a}\right)dy$$

Integrating on both sides

$$\int z\,dz = \int\left(\sqrt{x^2+a}\right)dx + \int\left(\sqrt{y^2-a}\right)dy$$

$$\frac{z^2}{2} = \frac{x}{2}\sqrt{x^2+a} + \frac{a}{2}\log\left|x+\sqrt{x^2+a}\right| + \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\log\left|y+\sqrt{y^2-a}\right| + b$$

$$z = \left[x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\log\left|\frac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}}\right| + b\right]^{1/2}$$

where a and b are arbitrary constants

(IV) Clairaut's form

If the eq^n of the form $z = px + qy + f(p,q)$

For C.I., put $p = a$ and $q = b$ in (1),

$z = ax + by + f(a,b)$ which is complete integral of (1) and a, b are arbitrary constants.

→ Find the singular solution of P.D.E.

$$z = px + qy + \log pq \quad \text{--}(1)$$

The given P.D.E. (1) is in Clairaut's form, put

$p = a$ and $q = b$ in (1), we have

$z = ax + by + \log ab$ where arbitrary constants. --(2)

$$\frac{\partial z}{\partial a} = 0 \qquad\qquad \frac{\partial z}{\partial b} = y + \frac{1}{b} = 0 \quad \text{--}(4)$$

$$x + \frac{b}{ab} = 0$$

$$x + \frac{1}{a} = 0 \quad \text{--}(3) \qquad\qquad a = \frac{-1}{x},\ b = \frac{-1}{y}$$

Substituting values of a and b in (2), we have

$$z = \left(\frac{-1}{x}\right)x + \left(\frac{-1}{y}\right)y + \log\left(\frac{-1}{x}\cdot\frac{-1}{y}\right)$$

$$z = -2 - \log(xy)$$ which is req. singular solution.

Prove that the complete integral of the eqⁿ $(xp+yq-z)^2 = 1+p^2+q^2$ is

$$\Rightarrow xp+yq-z = \pm(1+p^2+q^2)^{1/2}$$

$$z = xp+yq \mp (1+p^2+q^2)^{1/2} \quad \text{--- (1)}$$

which is in Clairaut's form.

The C.I. of (1) is given by

$$z = Ax+By \mp (1+A^2+B^2)^{1/2} \quad \text{--- (2)}$$

where A and B are arbitrary constants.

Choose $A = -\frac{a}{c}$, $B = -\frac{b}{c}$ and substitute in (2),

$$z = -\frac{ax}{c} - \frac{b}{c}y \mp \sqrt{1+\frac{a^2}{c^2}+\frac{b^2}{c^2}}$$

$$cz = -ax-by \mp \sqrt{a^2+b^2+c^2}$$

$$ax+by+cz = \mp\sqrt{a^2+b^2+c^2}$$

Taking positive sign, $ax+by+cz = \sqrt{a^2+b^2+c^2}$

Solve: $4xyz = pq + 2px^2y + 2qxy^2$ and also find its singular solution --- (1)

Let $x^2 = X$ and $y^2 = Y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}$$

$p = 2x \frac{\partial z}{\partial X}$

Similarly $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$q = 2y \frac{\partial z}{\partial Y}$

Putting values of p and q in (1), we have

$$4xyz = pq \left(2x\frac{\partial z}{\partial X}\right)\left(2y\frac{\partial z}{\partial Y}\right) + 2\left(2x\frac{\partial z}{\partial X}\right)x^2y + 2\left(2y\frac{\partial z}{\partial Y}\right)xy^2$$

$$Z = \frac{\partial z}{\partial X}\frac{\partial z}{\partial Y} + x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \quad \text{--- (2)}$$

Let $\frac{\partial z}{\partial X} = P$, $\frac{\partial z}{\partial Y} = Q$ --- (3)

using (3) in (2)

$$Z = PQ + PX + QY \quad \text{--- (4)},$$

which is in Clairaut's form, whose C.I. of (3) is

$Z = ab + aX + bY$ where a, b are arbitrary constants

Putting values of X and Y

$$z = ab + ax^2 + by^2 \quad \text{--- (5)}$$

For S.S,

$$\frac{\partial z}{\partial a} = b + x^2 = 0$$

$$\frac{\partial z}{\partial b} = a + y^2 = 0$$

Substituting the values of a and b in (5), we get the req. S.I

$$z = -x^2y^2 + x^2y^2 - x^2y^2 = -x^2y^2$$

$$z + x^2y^2 = 0$$