

Integral Equations

Part-II

By

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Solution of Volterra integral eqn of 2nd kind -

$$g(x) = f(x) + \lambda \int_a^x K(x,t) g(t) dt \quad \text{--- (1)}$$

then soln of (1) is

$$g(x) = f(x) + \lambda \int_a^x R(x,t,\lambda) f(t) dt$$

Q. With the help of resolvent kernel solve the following IE

i) $g(x) = e^{x^2} + \int_0^x e^{x^2-t^2} g(t) dt$

ii) $g(x) = 1 - 2x - \int_0^x e^{x^2-t^2} g(t) dt$

iii) Consider the volterra 2nd kind IE

$$g(x) = e^{x^2} + \int_0^x e^{x^2-t^2} g(t) dt. \quad \text{--- (1)}$$

On comparing eqn(1) with given IE, we have

$$f(x) = e^{x^2}, \quad K(x,t) = e^{x^2-t^2}$$

for iterated kernel

$$K_n(x,t) = K(x,t) = e^{x^2-t^2}$$

$$\& \quad K_n(x,t) = \int_t^x K(x,z) K_{n-1}(z,t) dz, \quad n=2,3,\dots$$

$$\therefore K_2(x,t) = \int_t^x K(x,z) K_1(z,t) dz$$

$$= \int_t^x e^{x^2-z^2} e^{z^2-t^2} dz$$

$$= \int_t^x e^{x^2-t^2} dz$$

$$K_2(x, t) = (x-t) e^{x^2-t^2}$$

$$\begin{aligned} K_3(x, t) &= \int_t^x K(x, z) K_2(z, t) dz \\ &= \int_t^x e^{x^2-z^2} (z-t) e^{z^2-t^2} dz \end{aligned}$$

$$\begin{aligned} K_3(x, t) &= e^{x^2-t^2} \left[(z-t)^2 \right]_t^x \\ &= \frac{e^{x^2-t^2}}{2} (x-t)^2 = \frac{e^{x^2-t^2}}{2!} (x-t)^2 \end{aligned}$$

$$\therefore K_m(x, t) = e^{x^2-t^2} \frac{(x-t)^{m-1}}{(m-1)!}, m=1, 2, 3, \dots$$

Therefore resolvent kernel is given by

$$\begin{aligned} R(x, t, \lambda) &:= \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x, t) \\ &= \sum_{m=1}^{\infty} \lambda^{m-1} e^{x^2-t^2} \frac{(x-t)^{m-1}}{(m-1)!} \\ &= e^{x^2-t^2} \sum_{m=1}^{\infty} \frac{(x-t)^{m-1}}{(m-1)!} \\ &= e^{x^2-t^2} \left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + \dots \right] \\ &= e^{x^2-t^2} e^{x-t} \end{aligned}$$

$$R(x, t, \lambda) = e^{(x-t)(x+t+1)}$$

Soln of given IE is

$$g(x) = f(x) + \int_0^x R(x, t, \lambda) f(t) dt$$

$$\begin{aligned}
 &= e^{x^2} + \int_0^x e^{(x-t)(x+t+1)} e^{t^2} dt \\
 &= e^{x^2} + \int_0^x e^{x^2-t^2} e^{x-t} e^{t^2} dt \\
 &= e^{x^2} + e^{x^2+x} (-e^{-t})_0^x \\
 &= e^{x^2} + e^{x^2+x} (-e^{-x} + e^{-0}) \\
 &= e^{x^2} - e^{x^2+x} - xe^{x^2+x} + e^{x^2+x} \\
 &= e^{x^2+x}
 \end{aligned}$$

* Soln of Fredholm IE of 2nd kind by successive substitution -

This Let $g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt$ — (1)

be given Fredholm IE of 2nd kind. Suppose that

i) Kernel $K(x,t) \neq 0$ is real & continuous in the rectangle R for which $a \leq x \leq b$, $a \leq t \leq b$. Suppose that $|K(x,t)| \leq P$, where P is the maximum value of $|K(x,t)|$ in R

ii) $f(x) \neq 0$ is real & continuous in an interval I: $a \leq x \leq b$. Let $|f(x)| \leq Q$ where Q is the max. value of $|f(x)|$ in I

iii) λ is a constant s.t. $|M| < \frac{1}{P(b-a)}$

Then (1) has a unique continuous soln in I & this soln is given by the absolutely & uniformly convergent series

$$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(x,t_1) f(t_1) dt_1 dt + \dots$$

Note - The above theorem proves the uniqueness of soln of Fredholm IE of 2nd kind

* Soln of Volterra IE of 2nd kind by successive substitution -

Th. Let $g(x) = f(x) + \lambda \int_a^x K(x,t) g(t) dt$ (1)

be given Volterra IE of 2nd kind.
Suppose that

i) Kernel $K(x,t) \neq 0$ is real & continuous in the rectangle R for which $a \leq x \leq b$, $a \leq t \leq b$. Suppose that $|K(x,t)| \leq P$ where P is the max. value of $|K(x,t)|$ in R .

ii) $f(x) \neq 0$ is real & conti. in an interval $I: a \leq x \leq b$. let $|f(x)| \leq Q$ where Q is the max. value of $|f(x)|$ in I .

iii) If λ is non-zero numerical parameter then (1) has a unique continuous soln in I & this soln is given by the absolutely & uniformly convergent series

$$g(x) = f(x) + \lambda \int_a^x K(x,t) f(t) dt + \lambda^2 \int_a^x K(x,t) \int_a^x K(x,t_1) f(t_1) dt_1 dt + \dots$$

* Solution of Fredholm IE of the 2nd kind by successive approximation.
Iterative method (Iterated Scheme)

Neumann's Series -

Consider Fredholm IE of 2nd kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt$$

By Neumann's series n th approximatn

$$g_n(x) = f(x) + \sum_{m=1}^n \lambda^m \int_a^b K_m(x,t) f(t) dt \quad (1)$$

as $n \rightarrow \infty$ then series (1) is called
Neumann's series

$$g(x) = \lim_{n \rightarrow \infty} g_n(x)$$

$$= f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t) f(t) dt$$

* Solⁿ of Fredholm IE by successive approximatⁿ when resolvent kernel can not be obtained in closed form -

Consider the fredholm IE of 2nd kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt \quad \text{--- (1)}$$

as zeroth order approximatⁿ

$$g_0(x) = f_0(x)$$

then nth approximatⁿ is given by

$$g_n(x) = f(x) + \lambda \int_a^b K(x,t) g_{n-1}(t) dt$$

- a. Solve non-homo. FIE of 2nd kind by the method of successive approxi.
to 3rd order

i) $g(x) = 2x + \lambda \int_0^1 (x+t) g(t) dt, \quad g_0(x) = 1$

ii) $g(x) = 1 + \lambda \int_0^1 (x+t) g(t) dt, \quad g_0(x) = 1$

- (i) Ans. Consider the FIE of 2nd kind

$$g(x) = f(x) + \lambda \int_0^1 K(x,t) g(t) dt$$

then nth order approximatⁿ soln is

$$g_n(x) = f(x) + \lambda \int_0^1 K(x,t) g_{n-1}(t) dt$$

$\therefore g_n(x) = 2x + \lambda \int_0^1 (x+t) g_{n-1}(t) dt$

Now, $n = 1$

$$\begin{aligned}
 g_1(x) &= 2x + 1 \int_0^1 (x+t) g_0(t) dt \\
 &= 2x + 1 \int_0^1 (x+t) dt \\
 &= 2x + 1 \left[xt + \frac{t^2}{2} \right]_0^1 \\
 g_1(x) &= 2x + 1 \left[x + \frac{1}{2} \right]
 \end{aligned}$$

For $n = 2$

$$\begin{aligned}
 g_2(x) &= 2x + 1 \int_0^1 (x+t) g_1(t) dt \\
 &= 2x + 1 \int_0^1 (x+t) \left[2x + 1 \left(x + \frac{1}{2} \right) \right] dt \\
 &= 2x + 1 \left[2 \int_0^1 (x^2 + xt) dt + 1 \int_0^1 \left(x + \frac{1}{2} \right) (x+t) dt \right] \\
 &= 2x + 1 \left[2 \left[x^2 t + \frac{xt^2}{2} \right]_0^1 + 1 \int_0^1 \left[x^2 + xt + \frac{x^2}{2} + \frac{xt^2}{2} \right] dt \right] \\
 &= 2x + 1 \left[2 \left(x^2 + \frac{x}{2} \right) + 1 \left[x^2 t + \frac{xt^2}{2} + \frac{xt}{2} + \frac{t^2}{4} \right]_0^1 \right] \\
 &= 2x + 1 \left[2x^2 + x + 1 \left(x^2 + \frac{x}{2} + \frac{x}{2} + \frac{1}{4} \right) \right] \\
 &= 2x + 1 (2x^2 + x) + 1/2 (x^2 + x + 1/4)
 \end{aligned}$$

For $n = 2$

$$\begin{aligned}
 g_2(x) &= 2x + 1 \int_0^1 (x+t) g_1(t) dt \\
 &= 2x + 1 \int_0^1 (x+t) \left[2t + 1 \left(t + \frac{1}{2} \right) \right] dt
 \end{aligned}$$

$$\begin{aligned}
 & 2x + \lambda \left[2 \int_0^1 (xt + t^2) dt + \lambda \int_0^1 (x+t)(t+1/2) dt \right] \\
 &= 2x + \lambda \left[2 \left(\frac{xt^2}{2} + \frac{t^3}{3} \right)_0^1 + \lambda \left(\frac{xt^2}{2} + \frac{xt}{2} + \frac{t^3}{3} + \frac{t^2}{4} \right)_0^1 \right] \\
 &= 2x + \lambda \left[2 \left(\frac{x}{2} + \frac{1}{3} \right) + \lambda \left(\frac{x}{2} + \frac{x}{2} + \frac{1}{3} + \frac{1}{4} \right) \right] \\
 &= 2x + \lambda \left(x + \frac{2}{3} \right) + \lambda^2 \left(x + \frac{7}{12} \right)
 \end{aligned}$$

Now for $n = 3$

$$\begin{aligned}
 g_3(x) &= 2x + \lambda \int_0^1 (x+t) g_2(t) dt \\
 &= 2x + \lambda \int_0^1 (x+t) \left[2t + \lambda \left(t + \frac{2}{3} \right) + \lambda^2 \left(t + \frac{7}{12} \right) \right] dt \\
 &= 2x + \lambda \left[2x \int_0^1 t dt + \right. \\
 &\quad \left. + \lambda \int_0^1 (x+t) t dt + \lambda \int_0^1 (x+t) \left(t + \frac{2}{3} \right) dt \right. \\
 &\quad \left. + \lambda^2 \int_0^1 (x+t) \left(t + \frac{7}{12} \right) dt \right] \\
 &= 2x + \lambda \left[2 \left(\frac{xt^2}{2} + \frac{t^3}{3} \right)_0^1 + \lambda \left(\frac{xt^2}{2} + \frac{2xt}{3} + \frac{t^3}{3} + \frac{2t^2}{3} \right)_0^1 \right. \\
 &\quad \left. + \lambda^2 \left(\frac{xt^2}{2} + \frac{7xt}{12} + \frac{t^3}{3} + \frac{7t^2}{12} \right)_0^1 \right] \\
 &= 2x + \lambda \left[\left(x + \frac{2}{3} \right) + \lambda \left(\frac{7x}{6} + \frac{2}{3} \right) + \lambda^2 \left(\frac{13x}{12} + \frac{15}{24} \right) \right]
 \end{aligned}$$

* Neumann's series for Volterra IE
of 2nd kind -

Consider the VIE of 2nd kind.

$$g(x) = f(x) + \lambda \int_a^x K(x,t) g(t) dt \quad \text{--- (1)}$$

then Neumann's series of (1) is given by

$$g(x) = \lim_{n \rightarrow \infty} g_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x,t) f(t) dt$$

Q. Find the Neumann series for soln of following IE

$$\text{i)} \quad g(x) = (1+x) + \lambda \int_0^x (x-t) g(t) dt$$

$$\text{ii)} \quad g(x) = 1 + \int_0^x xt \cdot g(t) dt$$

(i) Consider VIE of 2nd kind

$$g(x) = (1+x) + \lambda \int_0^x (x-t) g(t) dt \quad \text{--- (1)}$$

$$\text{here } f(x) = 1+x \quad \& \quad K(x,t) = (x-t)$$

$$\text{and } K_1(x,t) = K(x,t) = (x-t)$$

Neumann's series of eqn (1) is given by

$$\begin{aligned} g(x) &= f(x) + \lambda \int_0^x K_1(x,t) f(t) dt \\ &= (1+x) + \lambda \int_0^x (x-t)(1+t) dt \end{aligned}$$

$$= (1+x) + 1 \int_0^x (x+xt-t-t^2) dt$$

$$= (1+x) + 1 \left[xt + xt^2 - \frac{t^2}{2} - \frac{t^3}{3} \right]_0^x$$

$$= (1+x) + 1 \left[\frac{x^2}{2} + \frac{x^3}{3} - x^2 - \frac{x^3}{3} - 0 \right]$$

$$= (1+x) + 1 \left[\frac{x^2}{2} + \frac{x^3}{6} \right]$$

Now, for $n=2$

we have

$$K_2(x, t) = \int_t^x K(x, z) K_1(z, t) dz$$

$$= \int_t^x (x-z)(z-t) dz$$

$$K_2(x, t) = \int_t^x (x-z) \cdot (z-t) dz$$

integrating by parts

$$= \left[(x-z) \left[\frac{(z-t)^2}{2!} \right] - (-1) \left[\frac{(z-t)^3}{3 \cdot 2} \right] \right]_t^x$$

$$= \frac{1}{3!} (x-t)^3$$

$$\text{Similarly } K_3(x, t) = \int_t^x K(x, z) K_2(z, t) dz$$

$$= \int_t^x (x-z) (z-t)^3 dz = \frac{1}{3!} \int_t^x (x-z) (z-t)^3 dz$$

$$= \frac{1}{3!} \left[(x-z) \frac{(z-t)^4}{4!} - (-1) \frac{(z-t)^5}{5 \cdot 4} \right]_t^x$$

$$= \frac{1}{3!} \frac{(x-t)^5}{5 \cdot 4} = \frac{(x-t)^5}{5!}$$

therefore Neumann's series of the given volterra IE is

$$g(x) = \lim_{n \rightarrow \infty} g_n(x)$$

$$= f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^x K_m(x, t) f(t) dt$$

$$= (1+x) + \left[\lambda \int_0^x K_1(x, t) (1+t) dt \right]$$

$$+ \lambda^2 \int_0^x K_2(x, t) (1+t) dt + \lambda^3 \int_0^x K_3(x, t) (1+t) dt + \dots$$

$$= 1+x + \lambda \int_0^x (x-t) (1+t) dt + \lambda^2 \int_0^x \frac{(x-t)^3}{3!} (1+t) dt$$

$$+ \lambda^3 \int_0^x \frac{(x-t)^5}{5!} (1+t) dt + \dots$$

$$\begin{aligned}
 &= 1+x + \lambda \left[(1+t) \frac{(x-t)^2 - (1) \frac{(x-t)^3}{3 \cdot 2}}{2(-1)} \right]_0 \\
 &\quad + \frac{\lambda^2}{3!} \left[(1+t) \frac{(x-t)^4 - (1) \frac{(x-t)^5}{5 \cdot 4}}{(-1)4} \right]_0 x \\
 &\quad + \frac{\lambda^3}{5!} \left[(1+t) \frac{(x-t)^6 - (1) \frac{(x-t)^7}{7 \cdot 6}}{(-1)6} \right]_0 x + \dots \\
 \\
 &= 1+x + \lambda \left[\frac{x^2 + x^3}{2! 3!} \right] + \frac{\lambda^2}{3!} \left[\frac{x^4 + x^5}{4! 5!} \right] \\
 &\quad + \frac{\lambda^3}{5!} \left[\frac{x^6 + x^7}{6! 7!} \right] \\
 \\
 &= 1+x + \lambda \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right) + \lambda^2 \left(\frac{x^4}{4!} + \frac{x^5}{5!} \right) \\
 &\quad + \lambda^3 \left[\frac{x^6}{6!} + \frac{x^7}{7!} \right] + \dots
 \end{aligned}$$

Note-1 in above exmp if $\lambda = 1$ then
 $g(x) = e^x$

2) In the above exmp. $R(x, t, \lambda)$ is not in closed form therefore we can not use following formula

$$g(x) = f(x) + \lambda \int_0^x R(x, t, \lambda) f(t) dt$$

* 8 Using the method of successive approximation solve the IE

$$g(x) = x - \int_0^x (x-t) g(t) dt, g_0(x) = 0$$

An.

Consider the IE x

$$g(x) = f(x) + \lambda \int_a^x K(x,t) g(t) dt \quad \text{--- (1)}$$

& given IE

$$g(x) = x - \int_0^x (x-t) g(t) dt \quad \text{--- (2)}$$

by comparing eqn (1) & (2)

$$f(x) = x \quad \text{&} \quad K(x,t) = (x-t)$$

then n th approximation soln is

$$g_n(x) = x - \int_0^x (x-t) g_{n-1}(t) dt, \quad n=1,2,\dots$$

for $n=1$

$$\begin{aligned} g_1(x) &= x - \int_0^x (x-t) g_0(t) dt \\ &= x - \int_0^x (x-t) \cdot 0 dt = x \\ &= g_1(x) = x \end{aligned}$$

for $n=2$

$$\begin{aligned} g_2(x) &= x - \int_0^x (x-t) g_1(t) dt \\ &= x - \int_0^x (x-t) \frac{t}{1!} dt \\ &= x - \int_0^x (xt - t^2) dt \\ &= x - \left[\frac{xt^2}{2} - \frac{t^3}{3} \right]_0^x = x - \left[\frac{x^3}{2} - \frac{x^3}{3} \right] \end{aligned}$$

$$g_2(x) = x - \frac{x^3}{6}$$

$$\begin{aligned}
 & \text{for } n=3 \quad x \\
 & g_3(x) = x - \int_0^x (x-t) g_2(t) dt \\
 & = x - \int_0^x (x-t) \left(t - \frac{t^3}{6} \right) dt \\
 & \quad \text{I} \quad \text{II} \\
 & = x - \left[(x-t) \left(\frac{t^2}{2} - \frac{t^4}{6 \cdot 4} \right) \right] = (-1) \left(\frac{t^3}{3!} - \frac{t^5}{6 \cdot 5 \cdot 4} \right) \\
 & = x - \left[\frac{x^3 - x^5}{3! \cdot 6 \cdot 5 \cdot 4} \right] \\
 & = x - \frac{x^3}{3!} + \frac{x^5}{5!}
 \end{aligned}$$

4 Integral Eqs with symmetric kernel-

Note If kernel is symmetric in integral eqn then atleast one eigen value is always exist.

Fundamental Defn -

1) Square Integrable Function -

A fn $f(x)$ is said to be square integrable fn on $[a,b]$ if

$$\int_a^b |f(x)|^2 dx < \infty$$

2.)

L_2 - Function - A square integrable fn is said to be L_2 -fn i.e.

f is L_2 -fn on $[a, b]$ if

$$\int_a^b |f(x)|^2 dx < \infty$$

i) A kernel $K(x, t)$ is L_2 fn if

$$\int_a^b \int_a^b |K(x, t)|^2 dx dt < \infty, \quad \forall x \in [a, b], t \in [a, b]$$

ii) $\int_a^b |K(x, t)|^2 dx < \infty \quad \forall x \in [a, b]$

iii) $\int_a^b |K(x, t)|^2 dt < \infty \quad \forall t \in [a, b]$

* Inner Product of two functions -

Let f & g are two L_2 -fns of a real variable x , $a \leq x \leq b$ then inner product of f & g - defined as

$$(f, g) = \langle f, g \rangle = \int_a^b f(x) \bar{g}(x) dx$$

where $\bar{g}(x)$ is the complex conjugate of $g(x)$.

* Orthogonal Function

Two fns f & g are said to be orthogonal if their inner product is zero i.e.

$$(f, g) = 0 \Rightarrow \int_a^b f(x) \bar{g}(x) dx = 0, \quad \forall x \in [a, b]$$

If $f(x)$ & $g(x)$ are real valued fn then f & g are orthogonal if

$$\int_a^b f(x) g(x) dx = 0$$

* Normalized Fn -

A fn $g(x)$ is said to be normalized if $\|g(x)\| = 1$

where

$$\|g(x)\|^2 = (g, g) = \int_a^b g(x) \bar{g}(x) dx$$

$$= \int_a^b |g(x)|^2 dx$$

$$\|g(x)\| = (g, g)^{\frac{1}{2}} = \left[\int_a^b g(x) \bar{g}(x) dx \right]^{\frac{1}{2}}$$

$$= \left[\int_a^b |g(x)|^2 dx \right]^{\frac{1}{2}}$$

* Complex Hilbert Space - A linear space (or vector space) of infinite dimension with inner product (x, y) which is a complex no. is called a complex hilbert space if it satisfy the following three axioms

i) The definiteness axioms : /+ve property
 $(x, x) > 0$ for $x \neq 0$

ii) The linearity axiom -

$$(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y)$$

where α, β are arbitrary complex constants

iii) Symmetry axiom -

$$(x, y) = (\bar{y}, x)$$

where '-' represents complex conjugate

Note- Fredholm operator K defined on

$$Kg = \int_a^b K(x, t) g(t) dt$$

where $K(x, t)$ is kernel of Fredholm IE

Adjoint operator to K

$$\bar{K}g = \int_a^b \bar{K}(t, x) g(t) dt$$

Th. If K is Fredholm operator then

$$(Kg, h) = (g, \bar{K}h)$$

Proof We know that

$$(f, g) = \int_a^b f(x) \bar{g}(x) dx$$

$$\& Kg = \int_a^b K(x, t) g(t) dt$$

$$\text{LHS} - (Kg, h)$$

$$= \left(\int_a^b K(x, t) g(t) dt, h \right)$$

$$\int_a^b \bar{h}(x) \left[\int_a^b K(x,t) g(t) dt \right] dx$$

By changing order of integration

$$\int_a^b g(t) \left[\int_{x=a}^b K(x,t) \bar{h}(x) dx \right] dt$$

x replace by t & vice-versa

$$= \int_a^b g(x) \left[\int_a^b K(t,x) \bar{h}(t) dt \right] dx$$

$$= \int_a^b g(x) \left[\int_a^b \bar{K}(t,x) h(t) dt \right] dx$$

$$= \underset{\text{--- } \times}{\left(g, \int_a^b \bar{K}(t,x) h(t) dt \right)}$$

$$= (g, \bar{K}h)$$

Note - If K is symmetric kernel then

$$(Kg, h) = (g, Kh)$$

then $\textcircled{*}$ becomes

$$= \left(g, \int_a^b K(x,t) h(t) dt \right)$$

$$\Rightarrow (Kg, h) = (g, Kh)$$

* Orthonormal Set - A set of g_i s is said to be orthonormal if (g_i, g_j)

$$(g_i, g_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Th If a kernel is symmetric then all its iterated kernels are also symmetric
Proof Since $K(x, t)$ is symmetric kernel
 $K(x, t) = \bar{K}(t, x) \quad \dots \quad (1)$
 where \bar{K} is complex conjugate of K
 & iterated kernel $K_n(x, t)$, $n=1, 2, 3, \dots$, defined on

$$K_1(x, t) = K(x, t) \quad \dots \quad (2)$$

$$K_n(x, t) = \int_a^b K(x, z) K_{n-1}(z, t) dz \quad \dots \quad (3)$$

$n = 2, 3, \dots$

$$K_n(x, t) = \int_a^b K_{n-1}(x, z) K(z, t) dz \quad \dots \quad (4)$$

$n = 2, 3, \dots$

Clearly by (1) & (2)

$$K_1(x, t) = \bar{K}_1(t, x)$$

\therefore result is true for $n=1$

putting $n=2$ in (3), we have

$$\begin{aligned} K_2(x, t) &= \int_a^b K(x, z) K_1(z, t) dz \\ &= \int_a^b K(x, z) K(z, t) dz \quad [\text{by (2)}] \\ &= \int_a^b \bar{K}(z, x) \bar{K}(t, z) dz \quad [\text{by (1)}] \end{aligned}$$

$$= \int_a^b \bar{K}(t, z) \bar{K}(z, x) dz$$

$$= \int_a^b \bar{K}(t, z) \bar{K}_1(z, x) dz \quad [\text{by (2)}]$$

$$= \bar{K}_2(t, x)$$

$$\Rightarrow K_2(x, t) = \bar{K}_2(t, x)$$

$\Rightarrow K_2$ is symmetric

\therefore result is true for $n=2$

Let $K_n(x, t)$ is symmetric for $n=m$
 $\therefore K_m(x, t) = \bar{K}_m(t, x) \quad \text{--- (5)}$

Now, for $n=m+1$ then by eqn (3)

$$K_{m+1}(x, t) = \int_a^b K(x, z) K_m(z, t) dz$$

$$= \int_a^b K(x, z) \bar{K}_m(t, z) dz \quad [\text{by eqn (5)}]$$

$$= \int_a^b \bar{K}(z, x) \bar{K}_m(t, z) dz \quad [\text{by (1)}]$$

$$= \int_a^b \bar{K}_m(t, z) \bar{K}(z, x) dz$$

$$= \bar{K}_{m+1}(t, x) \quad [\text{by (4)}]$$

$$[\bar{K}_n(x, t) = \int_a^b \bar{K}_{n-1}(x, z) \bar{K}(z, t) dz]$$

$$\bar{K}_{m+1}(x, t) = \int_a^b \bar{K}_m(x, z) \bar{K}(z, t) dz$$

$x \leftrightarrow t$

$$\bar{K}_{m+1}(x, t) = \int_a^b \bar{K}_m(t, z) \bar{K}(z, x) dz \quad]$$

$\therefore K_m(x, t)$ is symmetric for $n = m+1$
Hence by principle of mathematical induction, the result is true
for $n \in \mathbb{Z}^+$

Thus, iterated kernel are symmetric

→ Definitions -

Let $\langle f_n(x) \rangle$ be a sequence of f^n , then the seqn $\langle f_n \rangle$ converges uniformly on interval I if

$$\sup |f_m(x) - f_n(x)| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

→ The sequence $\langle f_n(x) \rangle$ converges uniformly to $f(x)$ if $\sup |f(x) - f_n(x)| \rightarrow 0$ as $n \rightarrow \infty$

Th: The set of eigen values of 2nd iterated kernel coincide with the set of squares of the eigen values of the given kernel

Proof: Let λ be an eigen value of kernel $K(x, t)$ & corresponding eigen fn is $g(x)$ then b

$$g(x) = \lambda \int_a^b K(x, t) g(t) dt \quad \dots \textcircled{1}$$

$$\Rightarrow g(x) = \lambda K g \quad \dots \textcircled{2}$$

where K is Fredholm operator

$\Rightarrow (I - \lambda K) g(x) = 0$, when I is the identity operator

$$\Rightarrow (I + \lambda K)(I - \lambda K) g(x) = (I + \lambda K) \cdot 0 \\ = (I^2 - \lambda^2 K^2) g(x) = 0$$

$$\Rightarrow (I - \lambda^2 K^2) g(x) = 0 \quad [\because I^2 = I] \\ = I g(x) - \lambda^2 K^2 g(x) = 0 \\ = g(x) - \lambda^2 \int_a^b K_2(x, t) g(t) dt$$

\Rightarrow Eigenvalue of 2nd iterated kernel $K_2(x, t)$ is $\underline{\lambda^2}$

Converse - let λ^2 be the eigen values of 2nd iterated kernel $K_2(x, t)$ therefore,

$$g(x) = \lambda^2 \int_a^b K_2(x, t) g(t) dt$$

$$\Rightarrow (I - \lambda^2 K^2) g(x) = 0 \\ = (I^2 - \lambda^2 K^2) g(x) = 0 \\ = (I + \lambda K)(I - \lambda K) g(x) = 0$$

If λ is an eigen value of kernel $K(x, t)$ then given property verified by default.

If λ is not an eigen value of $K(x, t)$ let $(I + \lambda K) g(x) = g_1(x)$ — (2)
Then by (1)

$$(I - \lambda K) g_1(x) = 0$$

Since λ is not an eigen value of $K(x, t)$, therefore

$$g_1(x) = 0$$

$$\Rightarrow (I + \lambda x) g(x) = 0 \quad [\text{by (2)}]$$

$$\Rightarrow g(x) = -\lambda \int_a^b K(x, t) g(t) dt$$

$\Rightarrow -\lambda$ is an eigen value of $K(x, t)$
Hence $K(x, t)$ has eigen value λ

Remark-In the above theorem kernel need not be symmetric

ii) In general if λ is an eigen value of kernel $K(x, t)$ then λ^n is the eigen value of n th iterated kernel $K_n(x, t)$

Theo If $K(x, t)$ is real symmetric & continuous & $K(x, t) \neq 0$ then all the characteristic constants are real

Proof Let characteristic constant λ_0 is non-real, then $\lambda_0 = u_0 + i v_0$, $v_0 \neq 0$ (*)
The homogenous IE

$$g(x) = -\lambda_0 \int_a^b K(x, t) g(t) dt \quad \text{--- (1)}$$

has atleast one non-zero soln i.e.
 $h(x) \neq 0$

Then by (1)

$$h(x) = (u_0 + i v_0) \int_a^b K(x, t) h(t) dt \quad \text{--- (**)}$$

Let, if possible $h(x)$ is real then

$$h(x) = \mu_0 \int_a^b K(x,t) h(t) dt + i \nu_0 \int_a^b K(x,t) h(t) dt$$

by comparing real & imaginary parts
we have

$$h(x) = \mu_0 \int_a^b K(x,t) h(t) dt \quad (2)$$

$$\text{&} \quad 0 = \nu_0 \int_a^b K(x,t) h(t) dt \quad (3)$$

Then by $(*)$, $\nu_0 \neq 0$

$$\Rightarrow \int_a^b K(x,t) h(t) dt = 0 \quad (4)$$

By $(**)$ & (4)

$$h(x) = 0$$

which is contradiction to our assumption that $h(x) \neq 0$
therefore $h(x)$ is not real

$$\text{i.e. } h(x) = p(x) + iq(x), \quad q(x) \neq 0$$

Now by $(**)$

$$\begin{aligned}
 p(x) + iq(x) &= (\mu_0 + i\nu_0) \int_a^b K(x,t) [p(t) + iq(t)] dt \\
 &= \mu_0 \int_a^b K(x,t) p(t) dt - \nu_0 \int_a^b K(x,t) q(t) dt \\
 &\quad + i \left[\nu_0 \int_a^b K(x,t) p(t) dt + \mu_0 \int_a^b K(x,t) q(t) dt \right]
 \end{aligned}$$

(5)

By comparing real & imaginary parts

$$p(x) = \mu_0 \int_a^b K(x,t) p(t) dt - \nu_0 \int_a^b K(x,t) q(t) dt$$

(6)

$$q(x) = \nu_0 \int_a^b K(x,t) p(t) dt + \mu_0 \int_a^b K(x,t) q(t) dt$$

(7)

(6) + i(7), we have

$$p(x) - iq(x) = (\mu_0 - i\nu_0) \int_a^b K(x,t) [p(t) - iq(t)] dt$$

(8)

Clearly $\lambda_0 \neq \bar{\lambda}_0$, where $\bar{\lambda}_0 = \mu_0 - i\nu_0$

If $\lambda_0 \neq \bar{\lambda}_0$ then, their solns are orthogonal.

$$\begin{aligned}
 \text{i.e. } \int_a^b h(x) \bar{h}(x) dx &= 0 \\
 &= \int_a^b |h(x)|^2 dx = 0 \\
 &= \int_a^b [\{p(x)\}^2 + \{q(x)\}^2] dx = 0
 \end{aligned}$$

$$\Rightarrow p^2(x) + q^2(x) = 0$$

$$\Rightarrow p(x) = 0 \quad \& \quad q(x) = 0$$

$$\Rightarrow h(x) = p(x) + iq(x) = 0$$

which is again contradiction to fact that $h(x) \neq 0$

Therefore our assumption that λ is not real is wrong.
Hence λ_0 is real.

* Expansion of Eigen fns & Bilinear Form -

Let $K(x,t)$ be a non null (non-zero), symmetric kernel which has a finite or infinite no. of eigen values (always real & non-zero). Consider then eigen values in sequence

$$\lambda_1, \lambda_2, \lambda_3, \dots \quad \text{--- (1)}$$

in such a way that each eigen values is repeated as many times as its multiplicity

Further, we determinate denumerate these eigen values in the order that corresponding to their absolute value

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \dots \leq |\lambda_n| \leq \dots$$

let $g_1(x), g_2(x), \dots, g_n(x), \dots$
be the sequence of eigen fns

corresponding eigen values given by (1)
 All eigen fⁿ's are linearly independent
 corresponding to same eigen value.
 Since kernel is symmetric therefore
 each eigen fⁿ is orthonormalized

Suppose that kernel $K(x, t)$ has
 at-least one eigen value say λ_1 &
 corresponding eigen fⁿ is $g_1(x)$.
 Therefore second truncated symmetric
 kernel is

$$K^{(2)}(x, t) = K(x, t) - \frac{g_1(x) \bar{g}_1(t)}{\lambda_1}$$

which is non-zero & it will have also
 atleast one eigen value say λ_2 (if
 more than one eigen value, then
 choose smallest one) & corresponding
 eigen fⁿs $g_2(x)$. Then third truncated
 kernel is

$$\begin{aligned} K^{(3)}(x, t) &= K^{(2)}(x, t) - \frac{g_2(x) \bar{g}_2(t)}{\lambda_2} \\ &= K(x, t) - \sum_{i=1}^2 \frac{g_i(x) \bar{g}_i(t)}{\lambda_i} \end{aligned}$$

Q. Find the eigen value & eigen fⁿ of IE

$$g(x) = 1 \int_{-1}^1 (x+t) g(t) dt$$

Solⁿ The given IE can be written as

$$g(x) = 1 \int_{-1}^1 g(t) dt + 1 \int_{-1}^x g(t) dt$$

$$\text{let } \int_{-1}^1 g(t) dt = C_1 \quad \text{--- (2)}$$

$$\text{& } \int_{-1}^1 t g(t) dt = C_2 \quad \text{--- (3)}$$

Then (1) becomes

$$g(x) = \lambda x C_1 + \lambda C_2 \quad \text{--- (4)}$$

By eq. (2) & (4)

$$\begin{aligned} \int_{-1}^1 (\lambda t C_1 + \lambda C_2) dt &= C_1 \\ &= \lambda C_1 \int_{-1}^1 t dt + \lambda C_2 \int_{-1}^1 dt = C_1 \\ &= 0 + 2\lambda C_2 = C_1 \\ &= C_1 - 2\lambda C_2 = 0 \quad \text{--- (5)} \end{aligned}$$

Similarly by (3) & (4)

$$\begin{aligned} \int_{-1}^1 t [\lambda t C_1 + \lambda C_2] dt &= C_2 \\ &= \lambda C_1 \int_{-1}^1 t^2 dt + \lambda C_2 \int_{-1}^1 t dt = C_2 \\ &= 2\lambda C_1 \left[\frac{t^3}{3} \right]_0^1 = C_2 \\ &= \frac{2}{3} \lambda C_1 = C_2 \Rightarrow [2\lambda C_1 - 3C_2 = 0] \quad \text{--- (6)} \end{aligned}$$

For non-trivial sol'n. by (5) & (6)

$$\begin{vmatrix} 1 & -2\lambda \\ 2\lambda & -3 \end{vmatrix} = 0$$

$$-3 + 4\lambda^2 = 0 \Rightarrow \lambda^2 = \frac{3}{4}$$

$$\lambda = \pm \sqrt{\frac{3}{4}}/2$$

For $\lambda = \frac{\sqrt{3}}{2}$, then (5) & (6)

$$C_1 - \sqrt{3} C_2 = 0$$

$$\sqrt{3} C_1 - 3 C_2 = 0 \Rightarrow C_1 - \sqrt{3} C_2 = 0$$

$$\Rightarrow \boxed{C_1 = \sqrt{3} C_2}$$

\therefore by (4)

$$g(x) = \frac{\sqrt{3}}{2} x \cdot \sqrt{3} C_2 + \frac{\sqrt{3}}{2} C_2$$

$$= \frac{\sqrt{3} C_2}{2} (\sqrt{3} x + 1)$$

$\therefore g(x) = \sqrt{3} x + 1$ is the eigen-fn
for $\lambda = \frac{\sqrt{3}}{2}$

For $\lambda = -\frac{\sqrt{3}}{2}$ then by (5) & (6)

$$C_1 + \sqrt{3} C_2 = 0$$

$$\& -\sqrt{3} C_1 + 3 \frac{\sqrt{3}}{2} C_2 = 0 \Rightarrow -\sqrt{3} (C_1 - \sqrt{3} C_2) = 0$$

For $\lambda = -\frac{\sqrt{3}}{2}$ then by eqn (5) & (6)

$$C_1 + \sqrt{3} C_2 = 0$$

$$-\sqrt{3} C_1 - 3 C_2 = 0 \Rightarrow -\sqrt{3} (C_1 + \sqrt{3} C_2) = 0$$

$$\therefore C_1 + \sqrt{3} C_2 = 0$$

$$\Rightarrow \boxed{C_1 = -\sqrt{3} C_2}$$

\therefore by eqn (4)

$$g(x) = -\frac{\sqrt{3}}{2} x C_1 - \frac{\sqrt{3}}{2} C_2$$

$$= \frac{\sqrt{3}}{2} C_2 (\sqrt{3} x - 1) \Rightarrow g(x) = \sqrt{3} x - 1 \text{ is eigen fn}$$

Q. Find eigen value & eigen fⁿ of homo. IE

$$g(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt} \right) g(t) dt$$

$$\lambda_1 = \frac{1}{2} (17 + \sqrt{265})$$

$$\lambda_2 = \frac{1}{2} (17 - \sqrt{265})$$

Solⁿ

Given IE can be written as,

$$g(x) = \lambda x \int_1^2 t g(t) dt + \frac{1}{x} \int_1^2 \frac{1}{t} g(t) dt \quad \text{--- (1)}$$

$$g(x) = \lambda x c_1 + \frac{1}{x} c_2 \quad \text{--- (2)}$$

$$\text{where } c_1 = \int_1^2 t g(t) dt \quad \text{--- (3)}$$

$$c_2 = \int_1^2 \frac{1}{t} g(t) dt \quad \text{--- (4)}$$

by eqn (2) & (3)

$$\begin{aligned} c_1 &= \int_1^2 t \left[\lambda t c_1 + \frac{1}{t} c_2 \right] dt \\ &= \lambda c_1 \int_1^2 t^2 dt + \lambda c_2 \int_1^2 dt \\ &= \frac{\lambda c_1 [8-1]}{3} + \lambda c_2 (2-1) = \frac{7\lambda c_1 + \lambda c_2}{3} \end{aligned}$$

$$c_1 - \frac{7\lambda c_1}{3} = \lambda c_2 \Rightarrow (3-7\lambda) c_1 = 3\lambda c_2$$

$$(3-7\lambda) c_1 - 3\lambda c_2 = 0 \quad \text{--- (5)}$$

Similarly from (2) & (4)

$$c_2 = \int_1^2 \frac{1}{t} \left(\lambda t c_1 + \frac{1}{t} c_2 \right) dt$$

$$= \lambda C_1 \int_1^2 dt + \lambda C_2 \int_1^2 \frac{1}{t^2} dt = \lambda C_1 (2-1) + \lambda C_2 \left[-\frac{1}{2} + 1 \right]$$

$$C_2 = \lambda C_1 + \frac{\lambda C_2}{2} \Rightarrow C_2 - \frac{\lambda C_2}{2} = \lambda C_1$$

$$(2-\lambda)C_2 - 2\lambda C_1 = 0$$

(6)

For non-trivial soln, by (5) & (6).

$$\begin{vmatrix} 3-7\lambda & -3\lambda \\ -2\lambda & 2-\lambda \end{vmatrix} = 0$$

$$= (3-7\lambda)(2-\lambda) - 6\lambda^2 = 0$$

$$= \lambda^2 - 17\lambda + 6 = 0 \Rightarrow \boxed{\lambda = \frac{17 \pm \sqrt{265}}{2}}$$

For $\lambda = \frac{17 + \sqrt{265}}{2}$, by (5) & (6)

Th. Hilbert - Schmidt Theorem -

Statement - If $\phi(x)$ can be written in the form

$$\phi(x) = \int_a^b K(x, t) \cdot h(t) dt \quad \text{--- (1)}$$

where $K(x, t)$ is a symmetric L_2 -kernel & $h(t)$ is a L_2 -fn; then $\phi(x)$ can be expanded in an absolutely & uniformly convergent fourier series w.r.t. t. the orthonormal system of eigen fns $g_1(x), g_2(x), \dots, g_n(x)$ of the kernel $K(x, t)$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n g_n(x) \quad \text{--- (2)}$$

where $\phi_n = (\phi, g_n)$

The fourier coefficient ϕ_n of the fn $\phi(x)$ are related to the fourier coefficient h_n of fns $h(x)$ by the reln

$$\boxed{\phi_n = \frac{h_n}{\lambda_n}} \quad \& \quad h_n = (h, g_n)$$

where λ_n are eigen value of kernel $K(x, t)$

- * Schmidit's soln of non-homogenous FIE of 2nd kind -

Consider the non-homogeneous FIE

of 2nd kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt \quad (1)$$

where $K(x,t)$ is real, continuous & symmetric & λ is not an eigen value

Soln of (1) is given by

$$g(x) = f(x) + \lambda \sum_m \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \quad (2)$$

where $f_m = (f, \phi_m)$

$$f_m = \int_a^b f(t) \phi_m(t) dt$$

$$\text{&} C_m = \int_a^b g(x) \phi_m(x) dx$$

$$C_m = a_m + f_m$$

Case-I - Unique Soln

If $\lambda \neq \lambda_m$ then soln is unique & is given by the eqn (2)

Case-II - Infinitely many soln

Let $\lambda = \lambda_k$, where λ_k is the k^{th} eigen value & also let $f_k = 0$

$$\text{i.e. } \int_a^b f(t) \phi_k(t) dt = 0$$

then eqn (1) has infinitely many soln

& Solⁿ is given by

$$g(x) = f(x) + A \phi_K(x) + \sum_{m=1}^{\infty} \frac{f_m}{\lambda_m} \phi_m(x)$$

where ' ' implies that we should neglect $m = K$ in the summation.
& A is arbitrary constant.

Case-III let $\lambda = \lambda_K$ where λ_K is the K^{th} eigen value & also let $f_K \neq 0$ then (1) has no solⁿ.

Q.
$$g(x) = (x+1)^2 + \int_{-1}^1 (xt + x^2 t^2) g(t) dt$$

find solⁿ of above eqn using hilbert schmidit's theorem

solⁿ Given eqn can be written as

$$g(x) = (x+1)^2 + x \int_{-1}^1 t g(t) dt + x^2 \int_{-1}^1 t^2 g(t) dt$$

Consider non-homogenous FIE of 2nd kind

$$g(x) = f(x) + \lambda \int_{-1}^1 K(x, t) g(t) dt \quad (1)$$

On comparing given non-homogenous FIE with (1), we have

$$\begin{aligned} f(x) &= (x+1)^2, \lambda = 1, K(x, t) \\ &= xt + x^2 t^2 \end{aligned}$$

For finding eigen value & eigen fn
take homogenous FIE of 2nd kind

$$g(x) = \lambda \int_{-1}^1 (xt + x^2 t^2) g(t) dt \quad \text{--- (2)}$$

Eqn (2) can be written as

$$g(x) = \lambda x \int_{-1}^1 t g(t) dt + \lambda x^2 \int_{-1}^1 t^2 g(t) dt$$

$$g(x) = \lambda x c_1 + \lambda x^2 c_2 \quad \text{--- (3)}$$

$$\text{where } c_1 = \int_{-1}^1 t g(t) dt \quad \text{--- (4)}$$

$$\text{& } c_2 = \int_{-1}^1 t^2 g(t) dt \quad \text{--- (5)}$$

Now by (3) & (4)

$$\begin{aligned} c_1 &= \lambda \int_{-1}^1 t [t c_1 + t^2 c_2] dt \\ &= \lambda 2 c_1 \int_0^1 t^2 dt = \frac{2 c_1}{3} \lambda \end{aligned}$$

$$\left(1 - \frac{2\lambda}{3}\right) c_1 = 0 \quad \text{--- (6)}$$

Again by (3) & (5)

$$\begin{aligned} c_2 &= \lambda \int_{-1}^1 t^2 [c_1 t + c_2 t^2] dt \\ &= \lambda 2 c_2 \int_0^1 t^4 dt = \frac{2\lambda c_2}{5} \\ &= \left(1 - \frac{2\lambda}{5}\right) c_2 = 0 \end{aligned} \quad \text{--- (7)}$$

For non-trivial soln

$$\begin{vmatrix} 1 - \frac{2\lambda}{3} & 0 \\ 0 & 1 - \frac{2\lambda}{5} \end{vmatrix} = 0$$

$$\left(1 - \frac{2\lambda}{3}\right) \left(1 - \frac{2\lambda}{5}\right) = 0$$

$$\Rightarrow \lambda = \frac{3}{2}, \frac{5}{2}$$

$$\text{say } \lambda_1 = \frac{3}{2}, \lambda_2 = \frac{5}{2}$$

$$\text{Since } \lambda_i = \lambda_{j(i)}, i=1,2$$

Therefore soln of the given IE
is unique

Now for $\lambda = \lambda_1 = \frac{3}{2}$ then by (6)

$$0 \cdot C_1 = 0$$

$$= 0 = 0$$

$$\& \text{ by (7)} \Rightarrow \left(1 - \frac{3}{5}\right) C_2 = 0$$

$$\Rightarrow C_2 = 0$$

putting values of C_1, C_2 & $\lambda_1 = \frac{3}{2}$
in (3), we have

$$g(x) = C_1 \cdot \frac{3}{2} x$$

$$\Rightarrow g_1(x) = C_1 x$$

therefore $g_1(x) = x$ is the eigen
fn corresponding to eigen value
 $\lambda_1 = \frac{3}{2}$

Now for $\lambda = \lambda_2 = 5/2$ then by (6)

$$\left(1 - \frac{5}{3}\right) c_1 = 0 \Rightarrow c_1 = 0$$

$$\text{by (7)} \Rightarrow \left(1 - \frac{2}{5} \cdot \frac{5}{2}\right) c_2 = 0 \\ = 0 = 0$$

Putting values of c_1, c_2 & λ_2 in (3)

$$g_2(x) = \pm \frac{5}{2} x^2 c_2 = c_2' x^2 \\ = g_2(x) = c_2' x^2$$

$\Rightarrow g_2(x) = x^2$ is the eigen fn corresponding to eigen value $\lambda_2 = 5/2$

Now let orthonormalized fn corresponding to eigen fn's $g_1(x)$ & $g_2(x)$ respectively

$$\therefore \phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{x}{\sqrt{\int_{-1}^1 [g_1(x)]^2 dx}}$$

$$= \frac{x}{\sqrt{\int_{-1}^1 x^2 dx}} = \frac{x}{\sqrt{\frac{2}{3}}} = \frac{x}{\sqrt{\frac{2}{3}}} \quad (10)$$

$$\phi_1(x) = \frac{\sqrt{3}}{\sqrt{2}} x$$

$$\phi_2(x) = \frac{g_2(x)}{\sqrt{\int_{-1}^1 [g_2(x)]^2 dx}} = \frac{g_2(x)}{\sqrt{\int_{-1}^1 x^2 dx}}$$

$$= \frac{x^2}{\left[2 \int_0^1 x^4 dx \right]^{1/2}} = \sqrt{\frac{5}{2}} x^2$$

(11)

∴ Solⁿ of the given IE is

$$g(x) = f(x) + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - 1} \phi_m(x) \quad (*)$$

$$\text{where } f_1(x) = \int_{-1}^1 f(x) \phi_1(x) dx$$

$$= \frac{\sqrt{3}}{2} \int_{-1}^1 (x+1)^2 x dx$$

$$= \frac{\sqrt{6}}{2} \int_{-1}^1 (x^2 + 2x + 1) x dx$$

$$= \frac{\sqrt{6}}{2} \cdot 2 \times 2 \int_0^1 x^2 dx = \frac{2\sqrt{6}}{3}$$

$$f_2(x) = \int_{-1}^1 f(x) \phi_2(x) dx$$

$$= \frac{\sqrt{5}}{2} \int_{-1}^1 (x^2 + 2x + 1) x^2 dx$$

$$= \sqrt{\frac{5}{2}} \left[2 \int_0^1 x^4 dx + 2 \int_0^1 x^2 dx \right]$$

$$= \frac{\sqrt{10}}{2} \cdot 2 \left[\frac{1}{5} + \frac{1}{3} \right] = \sqrt{10} \cdot \frac{8}{15}$$

$$f_2(x) = \frac{8}{15} \sqrt{10}$$

(12)

Therefore (*) becomes

$$g_1(x) = (x+1)^2 + 1 \left[\frac{\frac{2\sqrt{6}}{3} - 1}{\frac{3}{2}} \cdot \frac{\sqrt{3}}{2} x + \frac{\frac{8\sqrt{10}}{15}}{\frac{5}{2}} \right]$$

$$g(x) = (x+1)^2 + 1 \left[\frac{2\sqrt{6}}{3} \times \frac{\sqrt{3}}{2} x + \frac{\frac{8\sqrt{10}}{15} \times \frac{5}{2}}{\frac{5}{2}-1} x^2 \right]$$

$$= (x+1)^2 + \left[\frac{2\sqrt{6}}{3} \times \frac{\sqrt{3}}{2} x \times 2 + \frac{8\sqrt{10} \times 2}{15} \times \frac{5}{3} x^2 \right]$$

$$= (x+1)^2 + \left[2 \left[\frac{6}{3} \times \sqrt{2} x + \frac{8}{15} \times \frac{2}{3} \times 5 x^2 \right] \right]$$

$$= (x+1)^2 + \left[4x + \frac{16}{9} x^2 \right]$$

$$= x^2 + 2x + 1 + 4x + \frac{16}{9} x^2$$

$$= \frac{25}{9} x^2 + 6x + 1$$

$$\Rightarrow g(x) = \frac{25}{9} x^2 + 6x + 1$$

Q. Solve the following symmetric IE with the help of hilbert schmidith theorem

$$g(x) = 1 + \lambda \int_0^{\pi} \cos(x+t) g(t) dt$$

Soln Consider the non-homogenous FIE of 2nd kind

$$g(x) = f(x) + \lambda \int_0^{\pi} K(x,t) g(t) dt$$

On comparing given eqn with above eqn we get

$$f(x) = 1, \quad K(x,t) = \cos(x+t), \quad \lambda' = \lambda$$

For finding eigen value & eigen fn consider homogenous FIE of 2nd kind

$$g(x) = \lambda'' \int_0^{\pi} \cos(x+t) g(t) dt \quad (1)$$

$$= \lambda'' \int_0^{\pi} (\cos x \cos t - \sin x \sin t) g(t) dt$$

$$= \lambda'' [\cos x \int_0^{\pi} \cos t g(t) dt - \sin x \int_0^{\pi} \sin t g(t) dt]$$

$$= \lambda'' [\cos x C_1 - \sin x C_2] \quad (2)$$

$$\text{where } C_1 = \int_0^{\pi} \cos t g(t) dt \quad (3)$$

$$\text{& } C_2 = \int_0^{\pi} \sin t g(t) dt \quad (4)$$

$$\begin{aligned}
 C_1 &= \int_0^{\pi} \cos t \{ \lambda'' (\cos t C_1 - \sin t C_2) \} dt \\
 &= \lambda'' C_1 \int_0^{\pi} \cos^2 t dt - \lambda'' C_2 \int_0^{\pi} \sin t \cos t dt \\
 &= \frac{\lambda'' C_1}{2} \int_0^{\pi} \frac{1 + \cos 2t}{2} dt - \frac{\lambda'' C_2}{2} \int_0^{\pi} \sin 2t dt
 \end{aligned}$$

$$C_1 = \frac{\lambda'' C_1}{2} (\pi) - \frac{\lambda'' C_2}{2} [-\cos 2\pi + \cos 0]$$

$$\lambda'' C_1 = \lambda'' C_1 (\pi)$$

$$(2 - \lambda'' \pi) C_1 = 0 \quad \text{--- (5)}$$

$$\begin{aligned}
 C_2 &= \int_0^{\pi} \sin t \{ \lambda'' \{ C_1 \cos t - C_2 \sin t \} \} dt \\
 &= C_1 \lambda'' \int_0^{\pi} \sin t \cos t dt - \lambda'' C_2 \int_0^{\pi} \sin^2 t dt \\
 &= \frac{C_1 \lambda''}{2} \int_0^{\pi} \sin 2t dt - \lambda'' C_2 \int_0^{\pi} \frac{(1 - \cos 2t)}{2} dt \\
 &= \frac{C_1 \lambda''}{2} [-\cos 2\pi + \cos 0] - \frac{\lambda'' C_2}{2} (\pi - 0)
 \end{aligned}$$

$$C_2 = -\frac{\lambda'' C_2}{2} \pi$$

$$(2 + \lambda'' \pi) C_2 = 0 \quad \text{--- (6)}$$

For non-trivial soln

$$\begin{vmatrix} 2 - \lambda'' \pi & 0 \\ 0 & 2 + \lambda'' \pi \end{vmatrix} = 0$$

$$= 4 - \lambda''^2 \pi^2 = 0$$

$$\lambda^2 = \frac{4}{\pi^2}$$

$$\lambda'' = \pm \frac{2}{\pi}$$

$$\lambda_1 = \frac{2}{\pi} \quad \& \quad \lambda_2 = -\frac{2}{\pi} \quad (\text{say})$$

Put value of λ_1 in (5) & (6)

$$\left(\frac{2 - 2 \times \pi}{\pi} \right) c_1 = 0$$

$$0 \cdot c_1 = 0$$

$$\left(\frac{2 + 2 \cdot \pi}{\pi} \right) c_2 = 0$$

$$\Rightarrow 4c_2 = 0 \Rightarrow c_2 = 0 \quad \text{--- (7)}$$

& $\lambda_2 = -\frac{2}{\pi}$ in (5) & (6)

$$\left(\frac{2 + 2 \cdot \pi}{\pi} \right) c_1 = 0$$

$$4c_1 = 0 \Rightarrow c_1 = 0 \quad \text{--- (8)}$$

$$\left(\frac{2 - 2 \cdot \pi}{\pi} \right) c_2 = 0$$

$$0 \cdot c_2 = 0$$

using $\lambda_1 = \frac{2}{\pi}$ & (7) in (2)

$$g_1(x) = \frac{2}{\pi} c_1 \cos x$$

$\Rightarrow g_1(x) = \cos x$ is the eigen fn corresponding to eigen value $\lambda_1 = \frac{2}{\pi}$

Similarly, $g(x) = -\frac{2}{\pi} C_2 \sin x$

$\Rightarrow g_2(x) = \sin x$ is eigen fn
corresponding to. $\lambda = -\frac{2}{\pi}$

Now orthonormalised fn corresponding
to eigen fn $g_1(x)$ & $g_2(x)$

$$\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{\cos x}{\left[\int_0^{\pi} \cos^2 x dx \right]^{1/2}}$$

$$= \frac{\cos x}{\left[\int_0^{\pi} 1 + \cos 2x dx \right]^{1/2}}$$

$$= \frac{\cos x}{\left[\frac{1}{2} (\pi - 0) \right]^{1/2}} = \frac{2}{\pi} \cos x$$

$$\phi_2(x) = \frac{g_2(x)}{\|g_2(x)\|} = \frac{\sin x}{\|\sin x\|}$$

$$= \frac{\sin x}{\left[\int_0^{\pi} \sin^2 x dx \right]^{1/2}} = \frac{2}{\pi} \sin x$$

Soln of given IE is

$$g(x) = f(x) + \lambda \sum_{m=1}^{\infty} \frac{f_m}{\lambda_m} \phi_m(x)$$

$$\text{Now, } f_1(x) = \int_0^{\pi} f(x) \phi_1(x) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \cos x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot [\sin x]_0^{\pi} = 0$$

$$\boxed{f_1(x) = 0}$$

$$f_2(x) = \int_0^{\pi} f(x) \phi_2(x) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin x \, dx = -\sqrt{\frac{2}{\pi}} [\cos x]_0^{\pi}$$

$$= -2 \sqrt{\frac{2}{\pi}} \neq 0$$

Case-I

If $\lambda \neq \lambda_1$ & $\lambda \neq \lambda_2$ then soln of given
IE is unique & given by

$$g(x) = f(x) + \lambda \sum_{m=1}^2 \frac{f_m}{\lambda_m - \lambda} \phi_m(x)$$

$$g(x) = 1 + \lambda \left[2 \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \sin x \right] \left(\frac{-2 - 1}{\pi} \right)$$

$$g(x) = 1 - \lambda \left(\frac{4 \sin x}{\pi} \right) \frac{(1+2/\pi)}{(1+2/\pi)}$$

Case-II

If $\lambda = \lambda_1$, $\lambda \neq \lambda_2$ & $f_1 = 0$

then infinitely many soln &
given by

$$g(x) = f(x) + A \phi_1(x) + \lambda \frac{f_2}{\lambda_2 - \lambda} \phi_2'(x)$$

$$= 1 + A \sqrt{\frac{2}{\pi}} \cos 2x + A \frac{\frac{2}{\pi} \sqrt{\frac{2}{\pi}}}{(-\frac{2}{\pi} - 1)} \times \sqrt{\frac{2}{\pi}} \sin x$$

III If $\lambda \neq \lambda_1$ & $\lambda = \lambda_2$
but $f_2 \neq 0$ no soln.

A Classical Fredholm Theory - Non-homogeneous FIE

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt$$

The Fredholm First Theorem -

The non-homogenous fredholm integral eqn of 2nd kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt \quad \text{--- (1)}$$

where the f 's $f(x)$ & $K(x,t)$ are integrable. Then (1) has unique soln & given by

$$g(x) = f(x) + \lambda \int_a^b R(x,t,\lambda) f(t) dt \quad \text{--- (2)}$$

where resultant kernel $R(x,t,\lambda)$ is meromorphic fn. of parameter λ defined by

[meromorphic fn - if singularity exist then it is only pole]

[Every analytic fn is meromorphic but converse need not be true]

$$R(x, t, \lambda) = \frac{D(x, t, \lambda)}{D(\lambda)} ; D(\lambda) \neq 0$$

where $D(x, t, \lambda)$ & $D(\lambda)$ are entire fn of parameter λ & defined by fredholm's series of the form

$$D(x, t, \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \int \dots \int$$

↓

$$K(z_1, z_2, \dots, z_m)$$

$$t, z_1, z_2, \dots, z_m$$

$$dz_1 dz_2 \dots dz_m$$

(4)

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \int \dots \int K(z_1, z_2, \dots, z_m) dz_1 dz_2 \dots dz_m$$

(5)

Both of which converges for all values of λ

where

$$K(x_1, x_2, \dots, x_n) = \begin{vmatrix} K(x_1, t_1) & K(x_1, t_2) & \dots & K(x_1, t_n) \\ K(x_2, t_1) & K(x_2, t_2) & \dots & K(x_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, t_1) & K(x_n, t_2) & \dots & K(x_n, t_n) \end{vmatrix}$$

is known as fredholm determinand

→ Soln of FIE -

$$g(x) = f(x) + \lambda \int_a^b R(x, t, \lambda) f(t) dt$$

(1)

$$R(x, t, \lambda) = \frac{D(x, t, \lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0 \quad (2)$$

where

$$D(x, t, \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t) \quad (3)$$

$$\& D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} u_m \quad (4)$$

In (3) & (4) coefficient $B_n(x, t)$ & $u_n(t)$ are given by

$$B_n(x, t) = \iiint_a^b K(x, t) \begin{vmatrix} K(x, z_1) & K(x, z_2) & \dots & K(x, z_n) \\ K(x_1, z_1) & K(x_1, z_2) & \dots & K(x_1, z_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, z_1) & K(x_n, z_2) & \dots & K(x_n, z_n) \end{vmatrix} dz_1 dz_2 \dots dz_n$$

$$K(z_1 z_2 \dots z_m) \leftarrow \begin{matrix} z_1 z_2 \dots z_n \end{matrix}$$

$$u_n(t) = \iiint_a^b \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) & \dots & K(z_1, z_n) \\ K(z_2, z_1) & K(z_2, z_2) & \dots & K(z_2, z_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(z_n, z_1) & K(z_n, z_2) & \dots & K(z_n, z_n) \end{vmatrix} dz_1 dz_2 \dots dz_n$$

where $D(x, t, \lambda)$ is known as Fredholm minor & $D(\lambda)$ is known as Fredholm determinant.

Q. Using Fredholm determinant find the resolvent kernel of following kernel

i) $K(x,t) = xe^t ; a=0, b=1$

Soln Here $K(x,t) = xe^t, a=0, b=1$
&

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)}, D(\lambda) \neq 0 \quad (2)$$

where

$$D(x,t,\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \int \dots \int K(x, z_1, z_2, \dots, z_m) dz_1 dz_2 \dots dz_m \quad (3)$$

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \int \dots \int K(z, z_1, z_2, \dots, z_m) dz dz_1 dz_2 \dots dz_m \quad (4)$$

$$\text{Now } B_1(x,t) = \int_0^1 K\left(\frac{x}{t}, z_1\right) dz_1$$

$$= \int_0^1 \begin{vmatrix} K(x,t) & K(x, z_1) \\ K(z, t) & K(z_1, z_1) \end{vmatrix} dz_1$$

$$= \int_0^1 \begin{vmatrix} xe^t & xe^{z_1} \\ z_1 e^t & z_1 e^{z_1} \end{vmatrix} dz_1$$

$$= \int_0^1 (xe^{z_1} e^{t+z_1} - xe^{z_1} e^{t+z_1}) dz_1 = 0$$

$$B_2(x,t) = \int_0^1 \int_0^1 K\left(\frac{x}{t}, z_1, z_2\right) dz_1 dz_2$$

$$= \int_0^1 \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) & K(x,z_2) \\ K(z_1,t) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,t) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1 dz_2$$

$$= \iint_0^1 \begin{vmatrix} \kappa e t & \kappa e^{z_1} & \kappa e^{z_2} \\ z_1 e t & z_1 e^{z_1} & z_1 e^{z_2} \\ z_2 e t & z_2 e^{z_1} & z_2 e^{z_2} \end{vmatrix} dz_1 dz_2$$

$$= \iint_0^1 \kappa z_1 z_2 \begin{vmatrix} e^t & e^{z_1} & e^{z_2} \\ e^t & e^{z_1} & e^{z_2} \\ e^t & e^{z_1} & e^{z_2} \end{vmatrix} dz_1 dz_2 = 0$$

$$\therefore B_m(x, t) = 0 \quad \forall m \neq 1$$

&

$$\begin{aligned} u_1 &= \int_0^1 K(z_1) dz_1 = \int_0^1 |K(z_1, z_1)| dz_1 \\ &= \int_0^1 z_1 e^{z_1} dz_1 \\ &= [z_1 e^{z_1} - (1)e^{z_1}]_0^1 \\ &= 1 \cdot e - 1 \cdot e + 1 = 1 \end{aligned}$$

$$u_2 = \iint_0^1 K \left(\frac{z_1}{z_1}, \frac{z_2}{z_2} \right) dz_1 dz_2$$

$$\begin{aligned} &= \iint_0^1 \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2 \\ &= \iint_0^1 \begin{vmatrix} z_1 e^{z_1} & z_1 e^{z_2} \\ z_2 e^{z_1} & z_2 e^{z_2} \end{vmatrix} dz_1 dz_2 \\ &= \iint_0^1 z_1 z_2 \begin{vmatrix} e^{z_1} & e^{z_2} \\ e^{z_1} & e^{z_2} \end{vmatrix} dz_1 dz_2 = 0 \end{aligned}$$

$$\therefore u_m = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{otherwise} \end{cases}$$

(5)

putting values in (3) & (4) from (5)
& (6), we have

$$\begin{aligned} D(x, t, \lambda) &= xe^{\lambda t} + 0 = xe^{\lambda t} \\ \text{&} D(1) &= 1 + \frac{(-1)}{1!} + 0 \\ &= \frac{1-1}{1} \end{aligned}$$

$$R(x, t, \lambda) = \boxed{\frac{xe^{\lambda t}}{1-1}}, \lambda \neq 1$$

- i) $R(x, t) = 2x - t \quad 0 \leq x \leq 1, 0 \leq t \leq 1$
- ii) $R(x, t) = \sin x \cos t \quad 0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi$
- iii) $R(x, t) = 1 + 3xt \quad 0 \leq x \leq 1, 0 \leq t \leq 1$
- iv) $R(x, t) = x^2t - xt^2 \quad 0 \leq x \leq 1, 0 \leq t \leq 1$

ii) Here $R(x, t) = \sin x \cos t, a=0, b=2\pi$
We know that, resolvent kernel \rightarrow ①

$$R(x, t, \lambda) = \frac{D(x, t, \lambda)}{D(1)} ; D(1) \neq 0 \quad ②$$

where

$$D(x, t, \lambda) = R(x, t) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} B_m(x, t) \quad ③$$

$$\text{&} D(1) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} u_m \quad ④$$

where $B_m(x, t) = \iiint \dots \int K \left[\begin{matrix} x, z_1, z_2, \dots, z_m \\ t, z_1, z_2, \dots, z_m \end{matrix} \right] dz_1 dz_2 \dots dz_m$

$$U_m = \iiint \dots \int K(z_1, z_2, \dots, z_m) dz_1 dz_2 \dots dz_m$$

$$\text{Now } B_1(x, t) = \int_0^{2\pi} K \begin{pmatrix} x & z_1 \\ t & z_1 \end{pmatrix} dz_1$$

$$= \int_0^{2\pi} \begin{vmatrix} K(x, t) & K(x, z_1) \\ K(z_1, t) & K(z_1, z_1) \end{vmatrix} dz_1$$

$$= \int_0^{2\pi} \begin{vmatrix} \sin x \cos t & \sin x \cos z_1 \\ \sin z_1 \cos t & \sin z_1 \cos z_1 \end{vmatrix} dz_1$$

$$= \int_0^{2\pi} \sin x \sin z_1 \begin{vmatrix} \cos t & \cos z_1 \\ \cos t & \cos z_1 \end{vmatrix} dz_1$$

$$= 0$$

$$\& B_2(x, t) = \iint \begin{vmatrix} K(x, t) & K(x, z_1) & K(x, z_2) \\ K(z_1, t) & K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, t) & K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2$$

$$= \iint \begin{vmatrix} \sin x \cos t & \sin x \cos z_1 & \sin x \cos z_2 \\ \sin z_1 \cos t & \sin z_1 \cos z_1 & \sin z_1 \cos z_2 \\ \sin z_2 \cos t & \sin z_2 \cos z_1 & \sin z_2 \cos z_2 \end{vmatrix} dz_1 dz_2$$

$$= 0$$

$$\therefore B_m(x, t) = 0 ; \forall m \geq 3$$

$$B_m(x, t) = 0 \quad \forall m \in \mathbb{Z}^+$$

$$\begin{aligned} \& \mu_1 = \int_0^{2\pi} |K(z_1, z_1)| dz_1 \\ &= \frac{1}{2} \int_0^{2\pi} 2 \sin z_1 \cos z_1 dz_1 \\ &= \frac{1}{2} \left[\frac{-\cos 2z_1}{2} \right]_0^{2\pi} = 0 \end{aligned}$$

$$\begin{aligned} \mu_2 &= \int_0^{2\pi} \int_0^{2\pi} K(z_1, z_2) dz_1 dz_2 \\ &= \int_0^{2\pi} \int_0^{2\pi} \begin{vmatrix} \sin z_1 \cos z_1 & \sin z_1 \cos z_2 \\ \sin z_2 \cos z_1 & \sin z_2 \cos z_2 \end{vmatrix} dz_1 dz_2 \end{aligned}$$

$$\begin{aligned} &= 0 \end{aligned}$$

$$\therefore \mu_m = 0 \quad \forall m \in \mathbb{Z}^+ \quad \text{--- (6)}$$

Using eqn (3) & (4), (5) & (6)

$$\begin{aligned} D(x, t, \lambda) &= K(x, t) + 0 \\ &= \sin x \cos t \end{aligned}$$

$$D(\lambda) = 1$$

$$R(x, t, \lambda) = \sin x \cos t$$

Alternate Approach (by recurrence relation)

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt \quad \text{--- (1)}$$

Soln of (1) is

$$g(x) = f(x) + \lambda \int_a^b R(x,t,\lambda) f(t) dt \quad \text{--- (2)}$$

where

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0$$

$$D(x,t,\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} B_m(x,t)$$

$$\& D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} u_m$$

For finding u_m & $B_m(x,t)$

$$u_0 = 1, \quad u_n = \int_a^b B_{n-1}(s,s) ds \quad \text{--- (3)}$$

$$\& B_0(x,t) = K(x,t)$$

$$B_n(x,t) = u_n K(x,t) - n \int_a^t K(x,z) B_{n-1}(z,t) dz, \quad n \geq 1$$

Note- The advantage of fredholm method is that resolvent kernel $R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)}$ is always uniformly convergent for all values of λ unless $D(\lambda) = 0$

Q: Using the recurrence relation find the resolvent kernel of following

$$(i) \quad K(x,t) = xe^{-xt}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1$$

$$K(x,t) = \sin xt \cos t \quad 0 \leq x \leq 2\pi, \quad 0 \leq t \leq 2\pi$$

$$K(x,t) = e^{x-t} \quad 0 \leq x; t \leq 1$$

$$K(x,t) = 4xt - x^2 ; \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1$$

Soln i) Hence $K(x,t) = x - \alpha t$, $\alpha = 0$, $b = 1$ — (1)

∴ We know that resolvent kernel

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0$$
 — (2)

where,

$$D(x,t,\lambda) = K(x;t) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} B_m(x,t)$$
 — (3)

$$\& D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} u_m$$
 — (4)

$$\text{let } u_0 = 1 \quad \& \quad B_0(x,t) = K(x,t) = x - \alpha t$$

$$u_n = \int_0^1 B_{n-1}(s,s) ds$$
 — (5)

$$\& B_n(x,t) = u_n K(x,t) - n \int_0^1 K(x,z) B_{n-1}(z,t) dz$$

$$n \geq 1$$
 — (6)

By (5)

$$u_1 = \int_0^1 B_0(s,s) ds$$

$$= \int_0^1 (s - 2s) ds = -\frac{1}{2}$$

by (6)

$$B_1(x,t) = u_1(x - \alpha t) - 1 \int_0^1 (x - 2z) B_0(z,t) dz$$

$$= -\frac{1}{2} (x - \alpha t) - \int_0^1 (x - \alpha z) (z - 2t) dz$$

$$= -\frac{1}{2} (x - \alpha t) - \left[\frac{(x - \alpha z)(z - 2t)^2}{2} - \frac{(-\alpha)(z - 2t)^3}{2 \cdot 3} \right]_0^1$$

$$= -\frac{1}{2}(x-2t) - \left[\frac{(x-2)(1-2t)^2}{2} + \frac{(1-2t)^3}{3} - x \cdot \frac{4t^2}{2} + \frac{8t^3}{3} \right]$$

$$= -\frac{1}{2}(x-2t) - \left[\frac{(x-2)(1+4t^2-4t)}{2} + \frac{(1-2t)^3}{3} - 2xt^2 + \frac{8t^3}{3} \right]$$

$$= -\frac{1}{2}(x-2t) - \left[\frac{x+2xt^2-2xt-1-4t^2+4t}{2} + \frac{1-8t^3-3x2t+12t^2}{3} - 2xt^2 + \frac{8t^3}{3} \right]$$

$$= -\frac{1}{2}(x-2t) - \left[\frac{x-2xt-1-4t^2+4t+1-2t+4t^2}{2} + \frac{1}{3} \right]$$

$$= -\frac{x+t}{2} - \frac{x+2xt}{2} + \frac{2}{3} - 2t$$

$$= \frac{2}{3} + 2xt - x - t$$

$$B_1(x,t) = \frac{2}{3} + 2xt - x - t$$

Again by (5)

$$M_2 = \int_0^1 B_1(s,s) ds$$

$$= \int_0^1 \left(\frac{2}{3} + 2s^2 - s - s \right) ds$$

$$= \frac{2}{3} + \frac{2}{3} - \frac{2}{2} = \frac{1}{3}$$

Again by (6)

$$B_2(x, t) = \mu_2 k(x, t) - 2 \int_0^1 k(x, z) B_1(z, t) dz$$

$$= \frac{1}{3} (x - 2t) - 2 \int_0^1 (x - 2z) \left(\frac{2}{3} + 2zt - z - t \right) dz$$

$$= \frac{1}{3} (x - 2t) - 2 \int_0^1 \left[\frac{2x}{3} + 2xzt - xz - tx \right.$$

$$\left. + 2z^2 - \frac{4}{3}z - 4z^2t + 2zt \right] dz$$

$$= \frac{1}{3} (x - 2t) - 2 \left[\frac{2x}{3} + 2xt \cdot \frac{1}{2} - x \cdot \frac{1}{2} - tx \right.$$

$$\left. \frac{2}{3} - \frac{4}{3} \cdot \frac{1}{2} - \frac{4t}{3} + 2t \cdot \frac{1}{2} \right]$$

$$= \frac{x - 2t}{3} - 2 \left(\frac{2x}{3} + xt - \frac{x}{2} - tx - \frac{2}{3} - \frac{1}{3}t + \frac{2}{3} \right)$$

$$= \frac{x - 2t}{3} - \frac{4}{3}x - 2xt + \cancel{\frac{2}{3}x} + 2xt + \cancel{\frac{4}{3}t} + \frac{2}{3}t - \cancel{\frac{4}{3}}$$

$$= 0$$

$$B_2(x, t) = 0$$

$$\therefore B_n(x, t) = 0 \quad \forall n \geq 2$$

$$\& \mu_n = 0 \quad n \geq 3$$

putting values of $B_m(x, t)$ & μ_m in (3) & (4) we have

$$D(x, t, \lambda) = (x - 2t) + \left[\frac{-1}{1!} B_1(x, t) + \frac{\lambda^2}{2!} B_2(x, t) \right]$$

$$= (x - 2t) + \left[-1 \left(\frac{2}{3} + 2xt - x - t \right) \right]$$

$$\begin{aligned} & \mathcal{D}(\lambda) = 1 + \left[-\frac{1}{1} u_1 + \frac{\lambda^2}{2} u_2 \right] \\ & = 1 + \left[\frac{1}{2} + \frac{\lambda^2}{6} \right] \end{aligned}$$

ii) Here $K(x,t) = \sin x \cos t$, $a=0$, $b=2\pi$ — (1)
We know that resolvent kernel

$$R(x,t,\lambda) = \frac{\mathcal{D}(x,t,\lambda)}{\mathcal{D}(\lambda)} = 1, \quad \mathcal{D}(\lambda) \neq 0 \quad — (2)$$

where

$$\mathcal{D}(x,t,\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \quad — (3)$$

$$\& \mathcal{D}(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} u_m \quad — (4)$$

let $u_0 = 1$ & $B_0(x,t) = K(x,t) = \sin x \cos t$

$$u_n = \int_0^{2\pi} B_{n-1}(s,s) ds \quad — (5)$$

$$\& B_n(x,t) = u_n K(x,t) - n \int_0^{2\pi} K(x,z) B_{n-1}(z,t) dz \quad n \geq 1 \quad — (6)$$

$$\begin{aligned} \text{by (5)} \quad u_1 &= \int_0^{2\pi} B_0(s,s) ds \\ &= \int_0^{2\pi} \sin s \cos s ds = \frac{1}{2} \int_0^{2\pi} \sin 2s ds \\ &= \frac{1}{2} \cdot \left[-\cos 2s + \cos 0 \right] = 0 \end{aligned}$$

by (6)

$$\begin{aligned}
 B_1(x, t) &= u_1 K(x, t) - \int_{-2\pi}^{2\pi} K(x, z) B_0(z, t) dz \\
 &= - \int_0^{2\pi} \sin x \cos z \sin z \cos t dz \\
 &= - \underbrace{\sin x \cos t}_{2} \int_0^{2\pi} \sin 2z dz = 0
 \end{aligned}$$

Since $B_1(x, t) = 0$

$$\therefore B_n(x, t) = 0 \quad \forall n \geq 1$$

$$\text{&} \quad u_n = 0 \quad \forall n \geq 1$$

putting values of $B_m(x, t)$ & u_n in eq.

(3) & (4) we have

$$\begin{aligned}
 D(x, t, \lambda) &= K(x, t) \\
 &= \sin x \cos t \\
 \text{&} \quad D(\lambda) &= 1
 \end{aligned}$$

$$R(x, t, \lambda) = \sin x \cos t$$

iii) Here $K(x, t) = e^{x-t}$; $a=0, b=1$

———— (1)

\therefore We know that resolvent kernel

$$R(x, t, \lambda) = \frac{D(x, t, \lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0$$

———— (2)

where

$$D(x, t, \lambda) = K(x, t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x, t)$$

———— (3)

$$\& D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} u_m \quad \text{--- (4)}$$

$$\text{let } u_0 = 1 \quad \& \quad B_0(x,t) := K(x,t) = e^{x-t}$$

$$u_n = \int_0^1 B_{n-1}(s,s) ds \quad \text{--- (5)}$$

$$\& B_n(x,t) = u_n K(x,t) - n \int_0^1 K(x,z) B_{n-1}(z,t) dz \quad n \geq 1 \quad \text{--- (5)}$$

$$\text{By eqn (5)} \quad u_1 = \int_0^1 B_0(s,s) ds = \int_0^1 e^{s-s} ds$$

$$u_1 = 1$$

$$\begin{aligned} \text{By eqn (6)} \quad B_1(x,t) &= u_1 K(x,t) - \int_0^1 e^{x-z} e^{z-t} dz \\ &= e^{x-t} - e^{x-t} \int_0^1 dz = 0 \end{aligned}$$

$$\text{since } B_1(x,t) = 0$$

$$\therefore B_n(x,t) = 0, \forall n \geq 1$$

$$u_n = 0 \quad \forall n \geq 2$$

putting these values of $B_m(x,t)$ & u_m in eqn (3) & (4), we have

$$D(x,t,\lambda) = e^{x-t} + 0 = e^{x-t}$$

$$\& D(\lambda) = 1 + \frac{(-\lambda)^1}{1!} = 1 - \lambda$$

$$\text{So } R(x,t,\lambda) = \frac{e^{x-t}}{1-\lambda}$$

iv) Here $K(x,t) = 4xt - x^2$; $a=0$, $b=1$ — (1)

We know that

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0 \quad — (2)$$

where

$$D(x,t,\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \quad — (3)$$

$$\text{&} D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} u_m \quad — (4)$$

$$\text{let } u_0 = 1 \quad \& \quad B_0(x,t) = K(x,t) = 4xt - x^2$$

$$u_n = \int_0^1 B_{n-1}(s,s) ds \quad — (5)$$

$$\text{&} B_n(x,t) = u_n K(x,t) - n \int_0^1 K(x,z) B_{n-1}(z,t) dz \quad n \geq 1 \quad — (6)$$

By eqn (5)

$$\begin{aligned} u_1 &= \int_0^1 B_0(s,s) ds \\ &= \int_0^1 (4s^2 - s^2) ds = 3 \left[\frac{s^3}{3} \right]_0^1 \end{aligned}$$

$$u_1 = 1$$

By eqn (6)

$$\begin{aligned} B_1(x,t) &= 4xt - x^2 - \int_0^1 (4xz - x^2)(4zt - z^2) dz \\ &= 4xt - x^2 - \int_0^1 (16xztz^2 - 4xz^3 - 4x^2t^2 + x^2z^2) dz \\ &= 4xt - x^2 - \left[16xt \cdot \frac{1}{3} - 4x \cdot \frac{1}{4} - 4x^2t \cdot \frac{1}{2} + x^2 \cdot \frac{1}{3} \right] \end{aligned}$$

$$4xt - x^2 - \frac{16}{3}xt + 2x + 2x^2t - \frac{x^2}{3}$$

$$= 2x^2t - \frac{4}{3}x^2 - \frac{4}{3}xt + 2x$$

Again By eq. (5)

$$M_2 = \int_0^1 B_1(s, s) ds = \int_0^1 \left(2s^3 - \frac{4}{3}s^2 - \frac{4}{3}s^2 + s \right) ds$$

$$= \frac{2}{4} - \frac{8}{3} \cdot \frac{1}{3} + \frac{1}{3} = 1 - \frac{8}{9} = \frac{1}{9}$$

Again by eqn. (6)

$$B_2(x, t) = \frac{1}{9} (4xt - x^2) - 2 \int_0^1 (4xz - x^2) \left[2z^2t - \frac{4}{3}z^2 \right] (-4zt + z) dz$$

$$= \left(\frac{4}{9}xt - \frac{x^2}{9} \right) - 2 \int_0^1 \left[8xtz^3 - \frac{16}{3}xz^3 - \frac{16}{3}xtz^2 + 4xz^2 \right. \\ \left. - 2x^2t z^2 + \frac{4}{3}x^2z^2 + \frac{4}{3}x^2t z - \frac{x^2z}{3} \right] dz$$

$$= \left(\frac{4}{9}xt - \frac{x^2}{9} \right) - 2 \left[\frac{8xt}{4} - \frac{16x}{3} \cdot \frac{1}{4} - \frac{16}{3}xt \cdot \frac{1}{3} + 4x \cdot \frac{1}{3} \right. \\ \left. - 2x^2t \cdot \frac{1}{3} + \frac{4}{3}x^2 \cdot \frac{1}{3} + \frac{4}{3}x^2t \cdot \frac{1}{2} - \frac{x^2}{2} \right]$$

$$= \left(\frac{4}{9}xt - \frac{x^2}{9} \right) - 4xt + \frac{32xt}{9} - \frac{8}{9}x^2 + x^2$$

$$-4xt + \frac{3\lambda}{9}xt + \frac{4}{9}xt + \frac{x^2}{9} - \frac{x^2}{9}$$

$$= 0$$

$$\text{since } B_2(x,t) = 0$$

$$B_n(x,t) = 0, \forall n \geq 2$$

$$M_n = 0, \forall n \geq 3$$

putting these values in eq. (3) & (4)

$$D(x,t,\lambda) = 4xt - x^2 + (-1)^1 \left(2x^2t - \frac{4}{3}x^2 - \frac{4}{3}xt \right. \\ \left. + x \right)$$

$$= 4xt - x^2 - 1 \left(2x^2t - \frac{4}{3}x^2 - \frac{4}{3}xt + x \right)$$

$$\& D(1) = 1 + (-1)^1 1 + \frac{(-1)^2 1}{2! 9} \\ = 1 + 1 + \frac{1^2}{18}$$

$$R(x,t,\lambda) = \frac{4xt - x^2 - 1 \left(2x^2t - \frac{4}{3}x^2 - \frac{4}{3}xt + x \right)}{1 - \lambda + \frac{1^2}{18}}$$