Integral Equations

Part-I

By

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Integral Eq^n - An eq^n is called integra eq^n if it contains integral sign at unknown function.

eg.:-

eg. $g(t) = f(t) + \int_{\Omega} K(x,t)\, g(x)\, dx$ (L.I.E.)

$g(t)$ → unknown function

known fn

here $K(x,t)$ = kernel

There are two types of integral eqⁿs

i) Linear Integral Eqn – An eqn is called L.I.E. if unknown function are operated by linear operator that is if only linear operatⁿs are perform on the unknown function that is unknown function ~~that~~ is linear.

Otherwise integral eqn

ii) called non-linear integral eqn.

eg.- $g(t) = f(t) + \int_{\Omega} K(x,t)\, g^2(x)\, dx$

Note- In integral eq., our purpose is that to find the unknown fn.

Solution

Solution of Integral Eqⁿ -
Solution of a I.E. is a functⁿ which when substituted in the I.E. reduces to an identity. that is if unknown fⁿ satisfy the given integral eqⁿ then the functⁿ is called solution of I.E.

* Some Kind of Integral Eqⁿ -
the general form of I.E. is

$$\alpha(x)\, g(x) = f(x) + \lambda \int_{\Omega} \underset{\text{kernel}}{K(x,t)}\ \underset{\text{unknown } f^n}{g(t)}\, dt$$

& $\alpha(x)$ & $f(x)$ are known as known functⁿ. $\lambda \in \mathbb{R}$ or $\mathbb{C}$
Ω = domain of the integral

The above is linear I.E. [power of unknown fⁿ must be 1]

1.) Voltera I.E. - An I.E. is called voltera I.E. if upper limit of integral is x & lower limit is constant that is

$$\alpha(x)\, g(x) = f(x) + \lambda \int_{a}^{x} K(x,t)\, g(t)\, dt \quad (1)$$

Special Cases -

i.) ~~First Kind~~ if $\alpha(x)$ in eq. (1) is non-constant that is function of x then the integral eq. (1) is called third kind of volterra eqn.

ii) If $\alpha(x) = 0$ in eq(1) then i. eq. (1) is called 1st kind volterra i. eq.

$$f(x) + \lambda \int_a^x K(x,t)\, g(t)\, dt = 0$$

(iii) If $\alpha(x) = 1$ in eq.(1) then eq. (1) is called 2nd kind volterra integral eq. i.e.

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)\, dt$$

if $f(x) = 0$ in eq. (1) then the volterra i. eq. is called homogenous V. i. eq. In particular if $\alpha(x) = 1$ & $f(x) = 0$ in eq. (1) then eq. is called 2nd kind volterra homo. i. eq.

2.) Fredhlomn Integral Eqn -

If the both limit of integral is contant in i. eq. (1) then the i. eq. is called fredhlomn integral. eq. i.e.

$$\alpha(x)\, g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

Special Cases -

i.) if $\alpha(x) = 0$ in eqⁿ(2), it is called 1st kind FIE.

$$0 = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

ii) if $\alpha(x) = 1$ in (2), it is called 2nd kind FIE

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

iii) if $\alpha(x)$ in eq (2) is not constant that is fⁿ of x then (2) is called 3rd kind FIE

<u>Note-</u> 1st & 2nd kind FIE is particular case of 3rd kind FIE.

if $f(x) = 0$ in eqⁿ (2) then the Fredholmn integral eqⁿ is called homogenous FIE.

In particular if if $\alpha(x) = 1$ & $f(x) = 0$ in eq (2) then it is called 2ⁿ kind fredhlomn homogenous integral eqⁿ.

→ Example of solutⁿ of integral eq.

Q. Show that $g(x) = (1+x^2)^{-3/2}$ is a solⁿ of voltera i. eq.

$$g(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2}\, g(t)\, dt$$

Ans. RHS $= \dfrac{1}{1+x^2} - \displaystyle\int_0^x \frac{t}{1+x^2}\, g(t)\, dt$

putting value of $g(t)$, we have

$$= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} (1+t^2)^{-3/2} dt$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^x \frac{t}{(1+t^2)^{3/2}} dt$$

put $1+t^2 = u^2$

$\Rightarrow 2t\,dt = 2u\,du$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_{t=0}^{t=x} \frac{u}{u^3} du =$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left[\frac{1}{(1+t^2)^{1/2}} \right]_0^x$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left[\frac{1}{(1+x^2)^{1/2}} - 1 \right]$$

$$= \frac{1}{(1+x^2)^{3/2}} = g(x)$$

therefore LHS = RHS

Hence $g(x)$ is the solun of given integral eq.

Q. Show that $g(x) = \frac{1}{\pi\sqrt{x}}$ is the soln of integral eq.

$$\int_0^x \frac{g(t)}{\sqrt{x-t}} dt = 1$$

(non-homo. 1st kind Voltera I.E.)

Ans. $\text{LHS} = \int_0^x \frac{g(t)}{\sqrt{x-t}} dt$

putting value of $g(t)$ in above eq.

$$\int_0^x \frac{1}{\pi\sqrt{t}} \cdot \frac{1}{\sqrt{x-t}}\,dt \quad = \cancel{\frac{1}{\pi}\int_0^x \frac{1}{t\sqrt{\frac{x}{t}-1}}}$$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{xt - t^2 + \frac{x^2}{4} - \frac{x^2}{4}}}\,dt$$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{\frac{x^2}{4} - \left(t^2 + \frac{x^2}{4} - xt\right)}}\,dt$$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - \left(t - \frac{x}{2}\right)^2}} \quad = \cancel{\frac{1}{\pi}\left[\frac{2}{x}\sin^{-1}\frac{t-x/2}{x/2}\right]_0^x}$$

$$\cancel{= \frac{2}{\pi x}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right] = \frac{2}{\pi x}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]}$$

$$\cancel{= \frac{2}{x}}$$

$$= \frac{1}{\pi}\left[\sin^{-1}\frac{t-x/2}{x/2}\right]_0^x \quad = \frac{1}{\pi}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] \quad = 1$$

Singular Integral Eqⁿ -

If limit of the integral in the integral eq. are infinite ~~or~~ (either lower or upper limit or both limit is ~~infinity~~ ∞) or kernal $K(x,t)$ becomes inf. ∞ at least one pt. lie in Ω, then the integral eq. is known as singular integral eqⁿ.

eg.- $\alpha(x)\, g(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x,t)\, g(t)\, dt$

Or $\alpha(x)\, g(x) = f(x) + \lambda \int_{0}^{\infty} K(x,t)\, g(t)\, dt$

$\alpha(x)\, g(x) = f(x) + \lambda \int_{-\infty}^{a} K(x,t)\, g(t)\, dt$

$g(x) = x + 2 \int_{-1}^{1} \underbrace{\frac{x}{t(t+2)}}_{\infty \text{ at } t=0} g(t)\, dt$

✱ Leibnitz's Rule -

$$\frac{d}{dx} \int_{g(x)}^{h(x)} F(x,t)\, dt = \int_{g(x)}^{h(x)} \frac{\partial \{F(x,t)\}}{\partial x} dt + F(x, h(x)) \frac{dh}{dx} - F(x, g(x)) \frac{dg}{dx}$$

→ In particular if $h(x)$ & $g(x)$ are constant

$$\frac{d}{dx} \int_{a}^{b} F(x,t)\, dt = \int_{a}^{b} \frac{\partial F}{\partial x} dt$$

✱ **Note** -

$$\underbrace{\int_a^x \int_a^x \int_a^x \cdots \int_a^x}_{(n \text{ times})} g(t)\, dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} g(t)\, dt$$

eg. - $\int_0^x \int_0^x \int_0^x f(t)\, dt = \int_0^x \frac{(x-t)^2}{2!} f(t)\, dt$

✱ → (upper limit must be in variable form)

Q. Verify whether the given f^n $g(x)$ is the sol^n of the volterra integral eq^n

$$g(x) = x - \frac{x^3}{6}, \quad g(x) = x - \int_0^x \sinh(x-t)\, g(t)\, dt$$

Ans. RHS - $x - \int_0^x \sinh(x-t)\, g(t)\, dt$

putting value of $g(t)$

$$= x - \int_0^x \sinh(x-t)\left(t - \frac{t^3}{6}\right) dt$$

$$\because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= x - \frac{1}{2}\int_0^x \left(e^{(x-t)} - e^{-(x-t)}\right)\left(t - \frac{t^3}{6}\right) dt$$

$$= x - \frac{1}{2} e^x \int_0^x \underset{\text{II}}{e^{-t}} \underset{\text{I}}{\left(t - \frac{t^3}{6}\right)} dt$$

$$+ \frac{1}{2} e^{-x} \int_0^x \underset{\text{II}}{e^{t}} \underset{\text{I}}{\left(t - \frac{t^3}{6}\right)} dt$$

$$= x - \frac{1}{2} e^x \left[\left(t - \frac{t^3}{6}\right)(-e^{-t}) - \left(1 - \frac{3t^2}{6}\right)(e^{-t})\right.$$

$$\cancel{+\frac{1}{2}e^{-x}} \left. \left(-\frac{6t}{6}\right)(-e^{-t}) - (-1)(e^{-t})\right]_0^x$$

$$+ \frac{1}{2} e^{-x} \left[\left(t - \frac{t^3}{6}\right)(e^t) - \left(1 - \frac{3t^2}{6}\right)(e^t)\right.$$

$$\left. + \left(-\frac{6t}{6}\right)(e^t) - (-1)\, e^t\right]_0^x$$

$$= x - \frac{1}{2} e^x \left[e^{-t}\left(t - \frac{t^3}{6} + t - 1 + 1 - \frac{3t^2}{6}\right)\right.$$

$$x-\frac{1}{2}e^{x}\left[\left(t-\frac{t^3}{6}\right)(-e^{-t})-\left(1-\frac{3t^2}{6}\right)e^{-t}+\left(\frac{-6t}{6}\right)(-e^{-t})\right.$$
$$\left.-(-1)e^{-t}\right]_0^x+\frac{1}{2}e^{-x}\left[\left(t-\frac{t^3}{6}\right)e^{t}\right.$$
$$\left.-\left(1-\frac{3t^2}{6}\right)e^{t}+\left(\frac{-6t}{6}\right)e^{t}-(-1)e^{t}\right]$$
$$=x-\frac{1}{2}e^{x}\left[e^{-t}\left(-t+\frac{t^3}{6}-1+\frac{t^2}{2}+t+1\right)\right]_0^x$$
$$+\frac{1}{2}e^{x}\left[e^{t}\left(t-\frac{t^3}{6}-1+\frac{t^2}{2}-t+1\right)\right]_0^x$$
$$=x-\frac{e^x}{2}\left[e^{-x}\left(\frac{x^3}{6}+\frac{x^2}{2}\right)-0\right]+\frac{e^{-x}}{2}\left[e^{x}\left(\frac{x^2}{2}-\frac{x^3}{6}\right)\right]$$
$$=x-\frac{1}{2}\left(\frac{x^3}{6}+\frac{x^2}{2}\right)+\frac{1}{2}\left(\frac{x^2}{2}-\frac{x^3}{6}\right)=x-\frac{x^3}{6}$$

Q. Verify whether the f^n $g(x)=\dfrac{1-2\sin x}{1-\pi/2}$ is the sol^n of i. eq.

$$g(x)-\int_0^{\pi}\cos(x+t)\,g(t)\,dt=1 \qquad [\text{no}]$$

An. LHS $= g(x)-\int_0^{\pi}\cos(x+t)\,g(t)\,dt$

$$=\frac{1-2\sin x}{1-\pi/2}-\int_0^{\pi}\cos(x+t)\left[\frac{1-2\sin t}{1-\pi/2}\right]dt$$
$$=\frac{1-2\sin x}{1-\pi/2}-\frac{1}{2}\int_0^{\pi}e^{(x+t)i}+e^{-(x+t)i}\left[\frac{1-2\sin x}{1-\pi/2}\right]dt$$
$$=\frac{1-2\sin x}{1-\pi/2}-\frac{1}{2}\left[\frac{e^{(x+t)i}}{(x+t)}+\frac{e^{-(x+t)i}}{(x+t)}\right]$$

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$$1 - \frac{2\sin x}{1-\pi/2} - \int_0^{\pi} \left\{ \cos(x+t) - \frac{2\sin x \cos(x+t)}{1-\pi/2} \right\} dt$$

$$LHS = \int_0^{\pi} \cos(x+t) g(t)\, dt = \int_0^{\pi} \cos(x+t) \left[1 - \frac{2\sin t}{(1-\pi/2)} \right] dt$$

$$= \int_0^{\pi} \cos(x+t)\, dt - \frac{2}{(1-\pi/2)} \int_0^{\pi} \cos(x+t) \sin t\, dt$$

$$= [\sin(x+t)]_0^{\pi} - \frac{2}{(1-\pi/2)} \int_0^{\pi} \left[\frac{\sin(x+2t) - \sin x}{2} \right] dt$$

$$= [\sin x \cos t + \cos x \sin t]_0^{\pi} - \frac{2}{(1-\pi/2)} \left[-\frac{\cos(x+2t)}{2} - \sin x\, t \right]_0^{\pi}$$

$$= -\sin x - \sin x + \frac{\cancel{2}}{(1-\pi/2)} \left[\frac{\cos x \cos 2t - \sin x \sin 2t}{2} + t \sin x \right]_0^{\pi}$$

$$= -2\sin x + \frac{\cancel{2}}{(1-\pi/2)} \left[\frac{\cos x}{2} + \pi \sin x - \frac{\cos x}{2} \right]$$

$$= -2\sin x + \frac{\pi \sin x}{(1-\pi/2)} = \frac{4(\pi-1)}{2-\pi} \sin x$$

$$\text{Now } 1 + \int_0^{\pi} \cos(x+t) g(t)\, dt = 1 + \frac{4(\pi-1)}{2-\pi} \sin x$$

$$\neq \cancel{\frac{-2+3\pi}{2-\pi}} \quad g(x)$$

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* Initial Value Problem
If in the diff. eqn conditn is given at only one pt.

eg. - $\frac{d^2y}{dx^2} + y = \sin x$, $y(0)=1$, $y'(0)=($

→ Boundry Value Problem -
If in the diff. eqn conditn at more than one pt.

Q. Convert the following diff. eqn into ~~Volterra~~ i. eq.

$\frac{d^2y}{dx^2} + y = 0$ when $y(0)=0$, $y'(0)=1$

Ans let $\frac{d^2y}{dx^2} = g(x)$ —— (1)

now integrating eqn (1) w.r.t. x from 0 to x, we have

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x g(x)\,dx$$

$$= \int_0^x g(t)\,dt$$

$$\frac{dy}{dx} - y'(0) = \int_0^x g(t)\,dt$$

$$\frac{dy}{dx} = 1 + \int_0^x g(t)\,dt \quad \text{—— (2)}$$

Again integrating w.r.t. x from 0 to x.

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$$[y(x)]_0^x = \int_0^x dx + \int_0^x\int_0^x g(t)\, dt^2$$

$$= y(x) - y(0) = x + \int_0^x \frac{(x-t)^1}{1!} g(t)\, dt$$

$$= \boxed{y(x) = x + \int_0^x (x-t)\, g(t)\, dt}$$

putting value of y & $\frac{d^2y}{dx^2}$ in given diff. eqn, we have

$$g(x) + x + \int_0^x (x-t)\, g(t)\, dt = 0$$

which is required integral eqn.
(Voltera int. eqn of 2nd kind with non-homo

Q. Form an int. eq. corresponding to the diff. eq.

$$\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x y = x$$

with IC $y(0) = 1$, $y'(0) = -1$

An. let $\frac{d^2y}{dx^2} = g(x)$ —— (1)

integrating eqn (1) w.r.t. x from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x g(t)\, dt$$

$$\frac{dy}{dx} - y'(0) = \int_0^x g(t)\, dt$$

$$\frac{dy}{dx} = -1 + \int_0^x g(t)\, dt \quad \text{—— (2)}$$

Again integrating eq.(2) w.r.t. x from 0 to x, we have

$$[y(x)]_0^x = -\int_0^x dx + \int_0^x\int_0^x g(t)\,dt^2$$

$$y(x) - y(0) = -x + \int_0^x (x-t)\,g(t)\,dt \quad \text{--- } \cancel{(3)}$$

$$y(x) = 1 - x + \int_0^x (x-t)\,g(t)\,dt \quad \text{--- } (3)$$

Ussing eq^n (1), (2) & (3) in given diff. eq.

$$\cancel{g(x) - \left[\sin x - 1 + \int_0^x g(t)\,dt\right]}$$

$$g(x) - \sin x\left[-1 + \int_0^x g(t)\,dt\right] + e^x\left[1 - x + \int_0^x (x-t)\,g(t)\,dt\right] = x$$

$$g(x) + \sin x - \sin x\int_0^x g(t)\,dt + e^x(1-x) + e^x\int_0^x (x-t)\,g(t)\,dt = x$$

$$= g(x) = x - \sin x - e^x(1-x) + \int_0^x \{\sin x - e^x(x-t)\}\,g(t)\,dt$$

which is required int. eq^n.

Q. Form an int. eq. corresponding to the diff. eq.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \text{with IC } y(0) = 1,\; y'(0) = 0$$

$$g(x) = -1 - \int_0^x (2x-t)\, g(t)\, dt$$

A. let $\frac{d^2y}{dx^2} = g(x)$ —— (1)

$\Rightarrow [y'(x)]_0^x = \int_0^x g(x)\, dx = \int_0^x g(t)\, dt$

$= \frac{dy}{dx} = \int_0^x g(t)\, dt$ —— (2)

again integrating above eqn w.r.t. x from 0 to x

$[y(x)]_0^x = \int_0^x \int_0^x g(t)\, dt^2$

$y(x) - 1 = \int_0^x (x-t)\, g(t)\, dt$

$y(x) = 1 + \int_0^x (x-t)\, g(t)\, dt$ —— (3)

Using eqn (1), (2) & (3) in given eqn

$g(x) + x\int_0^x g(t)\, dt + 1 + \int_0^x (x-t)\, g(t)\, dt = 0$

$g(x) = -1 + \int_0^x (-2x+t)\, g(t)\, dt$

$g(x) = -1 - \int_0^x (2x-t)\, g(t)\, dt$

which is the required eqn
(IInd kind volterra int. eqn with non-homogenous)

Q. Obtain ~~frehotf~~ fredhomn int. eq. of 2nd kind corresponding to the BVP

$$\frac{d^2y}{dx^2} + \lambda y = x \quad , \quad y(0) = 0 , \; y(1) = 1$$

Also recover the B.V.P. from int. eq. obtained.

Ans.

$$y(x) = \frac{5x}{6} + \frac{x^3}{6} + \lambda \int_0^1 K(x,t)\, y(t)\, dt$$

where $K(x,t) = \begin{cases} t(1-x) , & 0 \le t < x \\ x(1-t) & x \le t \le 1 \end{cases}$

Ans.

let $\frac{d^2y}{dx^2} = g(x)$

integrating above eqn w.r.t. x from 0 to 1

$$\left[\frac{dy}{dx}\right]_0^1 = \int_0^1 g(t)\, dt$$

Given eqn can be written as

$$\frac{d^2y}{dx^2} = x - \lambda y$$

integrating on both side from 0 to x, we have

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x x\, dx - \lambda \int_0^x y\, dx$$

$$\frac{dy}{dx} - y'(0) = \frac{x^2}{2} - \lambda \int_0^x y\, dx \quad \text{——} (2)$$

Again intergrating (2) w.r.t. x from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x x\,dx - \lambda\int_0^x y\,dx$$

$$\frac{dy}{dx} - y'(0) = \frac{x^2}{2} - \lambda\int_0^x y\,dx \quad \text{—— (2)}$$

$$y(x) - y(0) - xy'(0) = \frac{x^3}{6} - \lambda\int_0^x\int_0^x y\,dx^2$$

$$y(x) = xy'(0) + \frac{x^3}{6} - \lambda\int_0^x (x-t)\,y(t)\,dt$$

—— (3)

Given $y(1) = 1$

$$1 = y'(0) + \frac{1}{6} - \lambda\int_0^1 (1-t)\,y(t)\,dt$$

$$y'(0) = \frac{5}{6} + \lambda\int_0^1 (1-t)\,y(t)\,dt$$

putting value of $y'(0)$ in eq (3), we have

$$y(x) = \frac{5x}{6} + \lambda x\int_0^1 (1-t)\,y(t)\,dt + \frac{x^3}{6} - \lambda\int_0^x (x-t)\,y(t)\,dt$$

$$= \frac{5x}{6} + \frac{x^3}{6} + \lambda\int_0^1 (x-xt)\,y(t)\,dt - \lambda\int_0^x (x-t)\,y(t)\,dt$$

$$= \frac{5x}{6} + \frac{x^3}{6} + \lambda\int_0^x (x-xt)\,y(t)\,dt + \lambda\int_x^1 (x-xt)\,y(t)\,dt - \lambda\int_0^x (x-t)\,y(t)\,dt$$

$$= \frac{5x}{6} + \frac{x^3}{6} + \lambda \int_0^x (x - xt - x + t)\, y(t)\, dt$$

$$+ \lambda \int_x^1 x(1-t)\, y(t)\, dt$$

$$= \frac{5x}{6} + \frac{x^3}{6} + \lambda \int_0^x t(1-x)\, y(t)\, dt$$

$$+ \int_x^1 x(1-t)\, y(t)\, dt$$

$$y(x) = \frac{5x}{6} + \frac{x^3}{6} + \lambda \int_0^1 K(x,t)\, y(t)\, dt$$

where

$$K(x,t) = \begin{cases} t(1-x), & 0 \le t < x \\ x(1-t), & x \le t \le 1 \end{cases}$$

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Q. Convert the differential eqn

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0 \quad \text{with IC } y(0) = 1 \ \& \ y'(0) = 0$$

to a volteria int. eqn of 2nd kind. Conversely derive the original D.E. with I.C. from the int. eqn obtained

A.
$$\frac{d^2y}{dx^2} = 2x\frac{dy}{dx} + 3y \quad \text{—— (1)}$$

integrating above eqn w.r.t. x from 0 to x

$$\left[\frac{dy}{dx}\right]_0^x = 2\int_0^x x\frac{dy}{dx}\,dx + 3\int_0^x y\,dx$$

$$\frac{dy}{dx} - y'(0) = 2\left[\left(x\,y(x)\right)_0^x - (1)\int_0^x y\,dx\right] + 3\int_0^x y\,dx$$

$$\frac{dy}{dx} = 2x\,y(x) - 2\int_0^x y(t)\,dt + 3\int_0^x y(t)\,dt$$

$$= 2x\,y(x) + \int_0^x y(t)\,dt \quad \text{—— (2)}$$

Again integrating w.r.t. x b/w 0 to x

$$y(x) - y(0) = 2\int_0^x x\,y(x)\,dx + \int_0^x\int_0^x y(t)\,dt^2$$

$$y(x) = 1 + 2\int_0^x t\,y(t)\,dt + \int_0^x (x-t)\,y(t)\,dt$$

$$y(x) = 1 + \int_0^x (2t + x - t)\,y(t)\,dt$$

$$y(x) = 1 + \int_0^x (t+x)\, y(t)\, dt \quad \text{——— (3)}$$

which is required volterra i. eq. of 2nd kind.

Conversly - diff. (3) w.r.t. x on both sides, we have

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x (x+t)\, y(t)\, dt$$

$$= \int_0^x \frac{\partial}{\partial x}(x+t)\, y(t)\, dt + (x+x)\, y(x) \frac{d}{dx}(x) - (x+0)\, y(0) \frac{d}{dx}(0)$$

$$\frac{dy}{dx} = \int_0^x y(t)\, dt + 2x\, y(x) \quad \text{——— (4)}$$

put $x=0$ in (4) & (3)

$y'(0) = 0$, $y(0) = 1$

Again diff. eqn (4) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\int_0^x y(t)\, dt + 2\frac{d}{dx}[x\, y(x)]$$

$$= \int_0^x \frac{\partial}{\partial x} y(t)\, dt + y(x)\frac{d}{dx}(x) + 2y(x) + 2x\, dy/dx$$

$$\Rightarrow \frac{d^2y}{dx^2} = y(x) + 2y(x) + 2x\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 3y = 0$$

which is given DE with IC $y(0) = 1$, $y'(0) = 0$

☆ Solution of homogenous fredholmn int. eqn –

$$\gamma(x)\, g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt$$

→ Eigen value & Eigen function –

Consider the homogenous FIE of 2nd kind

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt \quad \text{—} ①$$

here $g(x) = 0$ is the trivial soln of (1) so parameter $\lambda \neq 0$ is called eigen value of given int. eq. (1) & corresponding $g(x) \neq 0$ is called eigen function

Remark - 1st - if $\lambda = 0$ in (1) then $g(x) = 0$ which is trivial soln therefore $\lambda = 0$ is not an eigen value

2.) A homogenous FIE may have no eigen value & eigen fn or it may not have any real eigen value & eigen fn.

3.) if $g(x)$ is eigen fn of (1) corresponding to eigen value λ_0 then $c\,g(x)$ where c is arbitotory constant is also eigen f^n corresponding to eigen value λ_0

4.) If the kernel $K(x,t)$ is continous in rectangle $R: a \le x \le b,\ a \le t \le b$ & no.s a & b are finite then to every eigen value λ there corresponds a finite no. of linearly independent eigen f^n. Then no. of such f^ns is called index of eigen value.

<u>Note</u>- Diff^n eigen values have diff^n indices

Q. Determine the eigen values & eigen f^ns for following integral eq^n for seperable kernel

* The kernel $K(x,t)$ is called seperable kernel if

$$K(x,t) = \sum_i f_i(x)\, g_i(t)$$

eg. - $K(x,t) = t(x + x^2 t)$
$= tx + t^2 x^2$

(i) $$g(x) = \lambda \int_{-1}^{1} (5xt^3 + 4x^2 t + 3xt)\, g(t)\, dt$$

(ii) $$g(x) = \lambda \int_0^1 (2xt - 4x^2)\, g(t)\, dt$$

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given i.eq. can be written as

$$g(x) = \cancel{\lambda t^3} \; \cancel{\lambda 5x \int_{-1}^{1} t^3 g(t)\,dt}$$

$$+ 4\lambda x^2 \int_{-1}^{1}$$

$$g(x) = \lambda \cdot 5x \int_{-1}^{1} t^3 g(t)\,dt + \lambda(4x^2+3x) \int_{-1}^{1} t\, g(t)\,dt \quad \text{---} (1)$$

let $\int_{-1}^{1} t^3 g(t)\,dt = C_1$ ——— (a)

$\int_{-1}^{1} t\, g(t)\,dt = C_2$ ——— (b)

then int. eq. (1) becomes

$$g(x) = 5\lambda C_1 x + \lambda(4x^2+3x) C_2 \quad \text{---} (2)$$

By (a) & (2)

$$\int_{-1}^{1} t^3 [5\lambda C_1 t + \lambda C_2 (4t^2+3t)]\,dt = C_1$$

$$= 5\lambda C_1 \int_{-1}^{1} t^4\,dt + 4\lambda C_2 \int_{-1}^{1} t^5\,dt + 3\lambda C_2 \int_{-1}^{1} t^4\,dt = C_1$$

$$= 5\lambda C_1 \cdot 2\int_{0}^{1} t^4\,dt + 0 + 3\lambda C_2 \cdot 2\int_{0}^{1} t^4\,dt = C_1$$

$$= 2\lambda C_1 + \frac{6\lambda C_2}{5} = C_1$$

$$= (10C_1 + 6C_2)\lambda = 5C_1$$

$$= (10\lambda - 5) C_1 + 6\lambda C_2 = 0 \quad \text{---} (3)$$

Similarly from eq. (b) & (2)

$$\int_{-1}^{1} t\,[5\lambda C_1 t + \lambda C_2 (4t^2+3t)]\,dt = C_2$$

$$= 5\lambda C_1 \int_{-1}^{1} t^2 dt + 4\lambda C_2 \int_{-1}^{1} t^3 dt + 3\lambda C_2 \int_{-1}^{1} t^2 dt = C_2$$

$$= 10\lambda C_1 \int_{0}^{1} t^2 dt + 0 + 6\lambda C_2 \int_{0}^{1} t^2 dt - C_2$$

$$= \frac{10\lambda C_1}{3} + 2\lambda C_2 = C_2$$

$$= 10\lambda C_1 + (6\lambda - 3) C_2 = 0 \quad \text{—— (4)}$$

For non-trivial sol^n, by (3) & (4) we have

$$\begin{vmatrix} 10\lambda - 5 & 6\lambda \\ 10\lambda & 6\lambda - 3 \end{vmatrix} = 0$$

$$(10\lambda-5)(6\lambda-3) - 60\lambda^2 = 0$$

$$60\lambda^2 - 30\lambda - 30\lambda + 15 - 60\lambda^2 = 0$$

$$60\lambda = 15 \Rightarrow \cancel{\lambda = \frac{60}{15}} = \cancel{4}$$

$$\lambda = \frac{15}{60} = \frac{1}{4}$$

Now put $\lambda = 1/4$ in eq (3) & (4) respec. we have

$$\left(\frac{10}{4} - 5\right) C_1 + \frac{6}{4} C_2 = 0$$

$$= -10 C_1 + 6C_2 = 0 = -5C_1 + 3C_2 = 0$$

$$= \boxed{5C_1 = 3C_2} \quad \text{—— (5)}$$

$$\& \quad \frac{10}{4} C_1 + \left(\frac{6}{4} - 3\right) C_2 = 0$$

$$10 C_1 - 6 C_2 = 0 \Rightarrow 5C_1 - 3C_2 = 0 \quad \text{—— (6)}$$

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putting value of λ & C_1 in (2), we have

$$g(x) = \frac{5}{4}\cdot\frac{3}{5} C_2 x + \frac{1}{4}(4x^2+3x) C_2$$

$$= \frac{3}{4} C_2 x + x^2 + \frac{3}{4} x C_2$$

$$= \frac{3}{2} C_2 x + x^2$$

$$g(x) = C_2\left(x^2 + \frac{3}{2}x\right)$$

$g(x) = x^2 + \frac{3x}{2}$ is the eigen fn corresponding to eigen value $\lambda = \frac{1}{4}$

(ii) $$g(x) = \lambda\int_0^1 (2xt - 4x^2)\, g(t)\, dt$$

Ans: given I.E. can be written as

$$g(x) = 2x\lambda\int_0^1 t\, g(t)\, dt - 4x^2\lambda\int_0^1 g(t)\,dt \quad \text{--- (1)}$$

let $$C_1 = \int_0^1 t\, g(t)\, dt \quad \text{--- (2)}$$

$$C_2 = \int_0^1 g(t)\, dt \quad \text{--- (3)}$$

then eqn (1) becomes

$$g(x) = 2x\lambda C_1 - 4x^2\lambda C_2 \quad \text{--- (4)}$$

From eqn (2) & (4)

$$C_1 = \int_0^1 t\,(2t\lambda C_1 - 4t^2\lambda C_2)\, dt$$

$$= 2\lambda C_1\int_0^1 t^2\, dt - 4\lambda C_2\int_0^1 t^3\, dt$$

$$C_1 = \frac{2\lambda C_1}{3} - \frac{4\lambda C_2}{4} = \frac{2\lambda C_1 - \lambda C_2}{3}$$

$$3C_1 = 2\lambda C_1 - 3\lambda C_2$$

$$= (2\lambda - 3) C_1 - 3\lambda C_2 = 0 \quad \text{——} \quad (5)$$

Similarly from eq. (3) & (4)

$$C_2 = \lambda \int_0^1 (2t C_1 - 4t^2 C_2)\, dt$$

$$C_2 = 2\lambda C_1 \int_0^1 t\, dt - 4\lambda C_2 \int_0^1 t^2 dt$$

$$C_2 = \lambda C_1 - \frac{4}{3}\lambda C_2$$

$$= 3C_2 = 3\lambda C_1 - 4\lambda C_2$$

$$= 3\lambda C_1 - (4\lambda + 3) C_2 = 0 \quad \text{——} \quad (6)$$

For non-trivial solⁿ, by eqⁿ (5) & (6)

$$\begin{vmatrix} 2\lambda - 3 & -3\lambda \\ 3\lambda & -(4\lambda+3) \end{vmatrix} = 0$$

$$-(4\lambda+3)(2\lambda-3) + 9\lambda^2 = 0$$

$$-[8\lambda^2 - 12\lambda + 6\lambda - 9] + 9\lambda^2 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda^2 + 3\lambda + 3\lambda + 9 = 0$$

$$(\lambda+3)(\lambda+3) = 0 \Rightarrow \boxed{\lambda = -3}$$

Put value of λ in eq. (5) & (6) respec.

$$= (-6-3) C_1 + 9 C_2 = 0$$

$$= 9C_1 = 9C_2 \Rightarrow \boxed{C_1 = C_2}$$

&

$$-9C_1 - (-12+3) C_2 = 0$$

$$= C_1 = C_2$$

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put value of λ & C_1 in eqn (4)

$$g(x) = 2x.(-3)C_1 - 4x^2(-3)C_1$$
$$= -6xC_1 + 12x^2C_1$$
$$= -6C_1(x-2x^2)$$

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Q. Show that homogenous I.E.

$$g(x) = \lambda \int_0^1 (t\sqrt{x} - x\sqrt{t})\, g(t)\, dt$$

does not have real eigen value & real eigen f^n

Ans. Given IE can be written as

$$g(x) = \lambda\sqrt{x}\int_0^1 t\, g(t)\, dt - \lambda x\int_0^1 \sqrt{t}\, g(t)\, dt \quad \text{--- (1)}$$

let $\int_0^1 t\, g(t)\, dt = C_1$ --- (2)

$\int_0^1 \sqrt{t}\, g(t)\, dt = C_2$ --- (3)

then eq. (1) becomes

$$g(x) = \lambda\sqrt{x}\, C_1 - \lambda x C_2 \quad \text{--- (4)}$$

by (2) & (4)

$$\int_0^1 t\, (\lambda\sqrt{t}\, C_1 - \lambda t C_2)\, dt = C_1$$

$$= \lambda C_1\int_0^1 t^{3/2}\, dt - \lambda C_2\int_0^1 t^2\, dt = C_1$$

$$= \frac{2}{5}\lambda C_1 - \frac{\lambda C_2}{3} = C_1$$

$$= 6\lambda C_1 - 5\lambda C_2 = 15 C_1$$

$$= (6\lambda - 15) C_1 - 5\lambda C_2 = 0 \quad \text{--- (5)}$$

by (3) & (4)

$$= \int_0^1 \sqrt{t}\, (\lambda\sqrt{t}\, C_1 - \lambda t C_2)\, dt = C_2$$

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$$\lambda C_1 \int_0^1 t\,dt - \lambda C_2 \int_0^1 t^{3/2}\,dt = C_2$$

$$= \frac{\lambda C_1}{2} - \lambda C_2 \frac{2}{5} = C_2$$

$$= 5\lambda C_1 - 4\lambda C_2 = 10 C_2$$

$$= \cancel{(5\lambda - 10) C_1 - 4\lambda C_2 = 0}$$

$$= 5\lambda C_1 - (4\lambda + 10) C_2 = 0 \quad \text{—— (6)}$$

For non-trivial solⁿ, by (5) & (6)

$$\begin{vmatrix} 6\lambda - 15 & -5\lambda \\ 5\lambda & -(4\lambda+10) \end{vmatrix} = 0$$

$$= -(6\lambda - 15)(4\lambda + 10) + 25\lambda^2 = 0$$

$$= -[24\lambda^2 + 60\lambda - 60\lambda - 50] + 25\lambda^2 = 0$$

$$= \lambda^2 + 50 = 0$$

$$= \lambda = \pm i\sqrt{50}$$

Since λ is complex therefore given I.E. can not have real eigen value & eigen fn.

Th. The characterstic no. of a symmetric kernel are real

★ Symmetric kernel - A kernel $K(x,t)$ is called symmetric kernel if

$$K(x,t) = \overline{K}(t,x)$$

where $\overline{K}$ is complex conjugate of K

Note - If $K(x,t)$ is real then for symmetric kernel

$$K(x,t) = K(t,x)$$

Proof Let homogenous FIE of second kind is

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt \quad \text{——— (1)}$$

(where λ is the eigen value & $g(x)$ is eigen f^n corresponding to λ)

let $\lambda_0 = \alpha + i\beta$ be complex eigen value of (1) & $g_0(x)$ be the eigen f^n corresponding to λ_0

then by eq^n (1)

$$g_0(x) = \lambda_0 \int_a^b K(x,t)\, g_0(t)\, dt \quad \text{——— (2)}$$

Since λ_0 is a complex eigen value of (1), therefore $\bar{\lambda}_0 = \alpha - i\beta$ is also an eigen value of (1) & $\bar{g}_0(x)$ be the eigen f^n corresponding to $\bar{\lambda}_0$

$$\therefore \bar{g}_0(x) = \bar{\lambda}_0 \int_a^b \bar{K}(x,t)\, \bar{g}_0(t)\, dt \quad \text{——— (3)}$$

Now multiplying (2) by $\bar{g}(x)$ & integrate w.r.t. x from a to b on both side, we have

$$\int_a^b g_0(x)\, \bar{g}_0(x)\, dx = \lambda_0 \int_a^b \bar{g}_0(x) \int_a^b K(x,t)\, g_0(t)\, dt\, dx$$

by change of order of integrat^n

$$= \lambda_0 \int_a^b g_0(t) \left[\int_a^b \bar{g}_0(x)\, K(x,t)\, dx \right] dt \quad \text{--- (4)}$$

In eqn(3) x replace by t, we have

$$\bar{g}_0(t) = \bar{\lambda}_0 \int_a^b \bar{K}(t,x)\, \bar{g}_0(x)\, dx \quad \text{--- (5)}$$

$\because$ $K(x,t)$ is symmetric kernel

$\therefore$ $K(x,t) = \bar{K}(t,x)$

then (5) becomes

$$\bar{g}(t) = \bar{\lambda}_0 \int_a^b K(x,t)\, \bar{g}(x)\, dx$$

$$\Rightarrow \int_a^b K(x,t)\, \bar{g}(x)\, dx = \frac{\bar{g}(t)}{\bar{\lambda}_0}$$

putting values in (4), we have

$$\int_a^b g(x)\, \bar{g}(x)\, dx = \lambda_0 \int_a^b g(t)\, \frac{\bar{g}(t)}{\bar{\lambda}_0}\, dt$$

$$\Rightarrow \bar{\lambda}_0 \int_a^b g(x)\, \bar{g}(x)\, dx = \lambda_0 \int_a^b g(x)\, \bar{g}(x)\, dx$$

$$= (\lambda_0 - \bar{\lambda}_0) \int_a^b g(x)\, \bar{g}(x)\, dx = 0$$

Since $g_0(x) \neq 0 \Rightarrow \bar{g}_0(x) \neq 0$

$$\Rightarrow g_0(x)\, \bar{g}_0(x) \neq 0$$

$$\Rightarrow \int_a^b g(x)\, \bar{g}(x)\, dx \neq 0$$

$$\therefore \lambda_0 - \bar{\lambda}_0 = 0$$

$$\Rightarrow \lambda_0 = \bar{\lambda}_0$$

$\Rightarrow \quad \alpha + i\beta = \alpha - i\beta$

$\Rightarrow \quad 2i\beta = 0$

$\Rightarrow \quad \beta = 0$

$\Rightarrow \quad \lambda_0 = \alpha \in R$

then eigen value of symmetric kernel are real.

☆ Orthogonality of Eigen F^n -

Th: The eigen functions of symmetric kernel, corresponding to diff^n eigenvalues are orthogonal.

Or

If $K(x,t)$ is symmetric kernel of homogenous integral eq^n of second kind

$$\phi(x) = \lambda \int_a^b K(x,t)\,\phi(t)\,dt$$

& $g_0(x)$ & $g_1(x)$ are eigen f^{n}s of $K(x,t)$ corresponding to eigen values λ_0 & λ_1 respec. $(\lambda_0 \neq \lambda_1)$, then $g_0(x)$ & $g_1(x)$ are orthogonal on interval $[a,b]$ i.e.

$$\int_a^b g_0(x)\,\bar{g}_1(x)\,dx = 0$$

Proof Let homogenous F.I.E. is

$$g(x) = \lambda \int_a^b K(x,t)\,g(t)\,dt \quad \text{——} \quad (1)$$

let λ_0 & λ_1 are two distinct eigen values of (1) & corresponding eigen f^n are $g_0(x)$ & $g_1(x)$ respectively.
then

$$g_0(x) = \lambda_0 \int_a^b K(x,t)\, g_0(t)\, dt \quad \text{——— (2)}$$

$$g_1(x) = \lambda_1 \int_a^b K(x,t)\, g_1(t)\, dt \quad \text{——— (3)}$$

Since kernel $K(x,t)$ is symmetric

$$\therefore \quad K(x,t) = \overline{K}(t,x) \quad \text{——— (4)}$$

Multiply (2) by $\bar{g}_1(x)$ & integrate w.r.t. x from a to b both side, we have

$$\int_a^b g_0(x)\, \bar{g}_1(x)\, dx = \lambda_0 \int_a^b \bar{g}_1(x) \left(\int_a^b K(x,t)\, g_0(t)\, dt \right) dx$$

By ~~charged~~ changing order of integratn

$$= \lambda_0 \int_a^b g_0(t) \int_a^b K(x,t)\, \bar{g}_1(x)\, dx\, dt \quad \text{——— (5)}$$

By eq (3)

$$g_1(t) = \lambda_1 \int_a^b K(t,x)\, g_1(x)\, dx$$

$$\Rightarrow \bar{g}_1(t) = \bar{\lambda}_1 \int_a^b \overline{K}(t,x)\, \bar{g}_1(x)\, dx$$

$$= \bar{\lambda}_1 \int_a^b K(x,t)\, \bar{g}_1(x)\, dx \qquad [\text{by eq } (4)]$$

$$= \lambda_1 \int_a^b K(x,t)\, \bar{g}_1(x)\, dx \qquad [\because \text{ eigen values of symmetric kernel are real.}]$$

$$\Rightarrow \int_a^b K(x,t)\, \bar{g}_1(x)\, dx = \frac{\bar{g}_1(t)}{\lambda_1} \quad \text{--- (6)}$$

by eq (5) & (6)

$$\lambda_1 \int_a^b g_0(x)\, \bar{g}_1(x)\, dx = \lambda_0 \int_a^b g_0(t)\, \bar{g}_1(t)\, dt$$

$$\Rightarrow (\lambda_1 - \lambda_0) \int_a^b g_0(x)\, \bar{g}_1(x)\, dx = 0$$

$\because \lambda_1 \neq \lambda_0 \Rightarrow \lambda_1 - \lambda_0 \neq 0$

$$\Rightarrow \int_a^b g_0(x)\, \bar{g}_1(x)\, dx = 0$$

$\therefore$ For symmetric kernel eigen fns corresponding two distinct eigen values are orthogonal.

☆ Eigen values for kernal in special form

Eg- Find the eigen value & eigen fn of the homo. I.E.

$$g(x) = \lambda \int_0^1 K(x,t)\, g(t)\, dt$$

where $K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1) & t \le x \le 1 \end{cases}$

Ans. the given I.E. can be written as

$$g(x) = \lambda \int_0^x K(x,t)\, g(t)\, dt + \int_x^1 K(x,t)\, g(t)\, dt$$

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Given $K(x,t) = \begin{cases} x(t-1) & , \quad 0 \le x \le t \\ t(x-1) & \quad t \le x \le 1 \end{cases}$

$(t \to x)$

$$g(x) = \lambda \int_0^x t(x-1)\, g(t)\, dt + \lambda \int_x^1 x(t-1)\, g(t)\, dt \quad \text{——— (1)}$$

Diff. eqn (1) w.r.t. x on both sides, we have

$$g'(x) = \lambda \frac{d}{dx} \int_0^x t(x-1)\, g(t)\, dt + \lambda \frac{d}{dx} \int_x^1 x(t-1)\, g(t)\, dt$$

by lebnitz rule

$$= \lambda \left[\int_0^x t\, g(t) \frac{\partial}{\partial x}(x-1)\, dt + x(x-1)\, g(x) \frac{d}{dx}(x) + \int_x^1 (t-1)\, g(t) \frac{\partial}{\partial x}(x)\, dt - x(x-1)\, g(x) \frac{d}{dx}(x) \right]$$

$$\Rightarrow g'(x) = \lambda \left[\int_0^x t\, g(t)\, dt + (x^2-x)\, g(x) + \int_x^1 (t-1)\, g(t)\, dt - (x^2-x)\, g(x) \right] \quad \text{——— (2)}$$

Again diff (2) w.r.t. x on both sides & applying lebnitz rule, we have

$$g''(x) = \lambda \left[x\, g(x) \frac{d}{dx}(x) - (x-1)\, g(x) \frac{d}{dx}(x) \right]$$

$$= g''(x) - \lambda g(x) = 0 \quad \text{——— (3)}$$

By eqn(1) $g(0) = 0$, $g(1) = 0$

Case-I If $\lambda = 0$, then by eqn (3)

$g''(x) = 0$

integrating above eqn w.r.t. x

$g'(x) = C_1$

Again integrating w.r.t. 'x'

$g(x) = C_1 x + C_2$

$g(0) = C_1 \cdot 0 + C_2 \Rightarrow C_2 = 0$

$\Rightarrow g(x) = C_1 x$

$g(1) = C_1$

$= C_1 = 0$

$\Rightarrow g(x) = 0$ which is trivial solⁿ

$\lambda = 0$ is not possible

Case-II If $\lambda = \mu^2 > 0$, $\mu \neq 0$

then by eqn (3)

$g''(x) - \mu^2 g(x) = 0$

Auxilliary eqⁿ - $m^2 - \mu^2 = 0$

$\Rightarrow m = \pm \mu$

$g(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$ —— (4)

$g(0) = 0$

$\Rightarrow 0 = C_1 + C_2$

& $g(1) = 0 \Rightarrow 0 = C_1 e^{\mu} + C_2 e^{-\mu}$

$= \begin{vmatrix} 1 & 1 \\ e^{\mu} & e^{-\mu} \end{vmatrix} = e^{-\mu} - e^{\mu} \neq 0$

because $\mu = 0$

Solution is unique

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$C_1 = C_2 = 0$

then by eq^n(4) , $g(x) = 0$
which is again trivial sol^n. Therefore
$\lambda > 0$ is not possible.

III If $\lambda = -\mu^2 < 0$ ~~($\mu>0$)~~ $(\mu \neq 0)$
then by eq^n(3)

$$g''(x) + \mu^2 g(x) = 0$$

Auxilary eq^n - $m^2 + \mu^2 = 0$

$$m = \pm \mu i$$

$$g(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

$$g(0) = 0$$

$$\Rightarrow 0 = C_1$$

$$\Rightarrow g(x) = C_2 \sin \mu x$$

$$\because g(1) = 0$$

$$\Rightarrow 0 = C_2 \sin \mu$$

$C_2 \neq 0$ [for non-trivial sol^n]

$$\sin \mu = 0$$

$$\sin \mu = \sin n\pi$$

$$\mu = n\pi \quad \forall n \in Z$$

$$\therefore \lambda = -\mu^2 = -n^2\pi^2$$

$$\& \quad g(x) = C_2 \sin(n\pi x) , \quad n \in Z$$

$g(x) = \sin n\pi x$ is the eigen function corresponding to eigen value $-n^2\pi^2$, $n \in Z$

Th. Solution of Fredholm I.E. of 2nd kind with seperable kernel [reduction of a system of Algebric eqn]

Proof Consider the fredholmn I.E. of 2nd kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt \quad — (1)$$

Since $K(x,t)$ is seperable kernel than

$$K(x,t) = \sum_i f_i(x)\, g_i(t) \quad — (2)$$

then by eqn (1) & (2)

$$g(x) = f(x) + \lambda \int_a^b \left[\sum_i f_i(x)\, g_i(t)\right] g(t)\, dt$$

$$= f(x) + \lambda \sum_i f_i(x) \underbrace{\int_a^b g_i(t)\, g(t)\, dt}_{C_i}$$

$$\therefore \; g(x) = f(x) + \lambda \sum_i C_i f_i(x)$$

where $C_i = \int_a^b g_i(t)\, g(t)\, dt$

Q. Solve

$$g(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \; g(t)\, dt$$

Ans. The given IE can be written as

$$g(x) = \sin x + \lambda \sin x \underbrace{\int_0^{\pi/2} \cos t \; g(t)\, dt}_{C}$$

[because lower limit & upper limit is constant So $C = \int_0^{\pi/2} \cos t \; g(t)\, dt$]

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$\Rightarrow \quad g(x) = \sin x + \lambda c \sin x$ —— ①

where $c = \int_0^{\pi/2} \cos t \, g(t) \, dt$

putting value of $g(t)$ by eqn (1) then we have

$$c = \int_0^{\pi/2} \cos t \, (1+\lambda c) \sin t \, dt$$

$$c = \left(\frac{\lambda c+1}{2}\right) \int_0^{\pi/2} 2 \sin t \cos t \, dt$$

$$= \left(\frac{\lambda c+1}{2}\right) \int_0^{\pi/2} \sin 2t \, dt$$

$$= \left(\frac{\lambda c+1}{2}\right) \left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$$

$$= -\frac{(\lambda c+1)}{2} \left[\cos \pi - \cos 0\right]$$

$$= -\frac{(\lambda c+1)}{4} \left[-1-1\right] = \frac{\lambda c+1}{2}$$

$$\Rightarrow \quad \lambda c + 1 = 2c$$

$$= \lambda c + 1 - 2c = 0$$

$$= c(\lambda - 2) + 1 = 0$$

$$\Rightarrow \quad c = \frac{-1}{\lambda - 2}, \quad \lambda \neq 2$$

putting this value in eq (1), we have

$$\Rightarrow \quad g(x) = \sin x - \frac{\lambda}{\lambda - 2} \sin x, \quad \lambda \neq 2$$

$$g(x) = \frac{\sin x (\lambda - 2) - \lambda \sin x}{\lambda - 2}$$

$$\boxed{g(x) = \frac{2\sin x}{2-\lambda}}, \quad \lambda \neq 0$$

Q. Solve the fredholm IE of 2nd kind

$$g(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)\, g(t)\, dt$$

Ans. The given IE can be written as

$$g(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)\, g(t)\, dt$$

$$g(x) = x + \lambda \int_0^1 xt^2 g(t)\, dt + \lambda \int_0^1 x^2 t\, g(t)\, dt$$

$$= x + \lambda x \int_0^1 t^2 g(t)\, dt + \lambda x^2 \int_0^1 t\, g(t)\, dt$$

$$g(x) = x + \lambda x C_1 + \lambda x^2 C_2 \quad \text{——— (1)}$$

where $C_1 = \int_0^1 t^2 g(t)\, dt$ ——— (2)

$$C_2 = \int_0^1 t\, g(t)\, dt \quad \text{——— (3)}$$

$$g(t) = t + \lambda t C_1 + \lambda t^2 C_2$$

$$C_1 = \int_0^1 t^2 [t + \lambda t C_1 + \lambda t^2 C_2]\, dt$$

$$= \int_0^1 t^3 dt + \lambda C_1 \int_0^1 t^3 dt + \lambda C_2 \int_0^1 t^4 dt$$

$$C_1 = \left[\frac{t^4}{4}\right]_0^1 + \lambda C_1 \left[\frac{t^4}{4}\right]_0^1 + \lambda C_2 \left[\frac{t^5}{5}\right]_0^1$$

$$C_1 = \frac{1}{4} + \frac{\lambda C_1}{4} + \frac{\lambda C_2}{5}$$

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$$20C_1 = 5 + 5\lambda C_1 + 4\lambda C_2$$

$$(20-5\lambda)C_1 - 4\lambda C_2 - 5 = 0 \quad \text{——— (4)}$$

From eqn(1) & eqn(3)

$$C_2 = \int_0^1 t\,[t + \lambda t C_1 + \lambda t^2 C_2]\,dt$$

$$C_2 = \int_0^1 t^2 dt + \lambda \int_0^1 t^2 C_1 dt + \lambda C_2 \int_0^1 t^3 dt$$

$$C_2 = \left[\frac{t^3}{3}\right]_0^1 + \lambda C_1 \left[\frac{t^3}{3}\right]_0^1 + \lambda C_2 \left[\frac{t^4}{4}\right]_0^1$$

$$C_2 = \frac{1}{3} + \lambda C_1 \cdot \frac{1}{3} + \lambda C_2 \cdot \frac{1}{4}$$

$$12C_2 = 4 + 4\lambda C_1 + 3\lambda C_2$$

$$(12-3\lambda)C_2 - 4\lambda C_1 - 4 = 0$$

$$(12-3\lambda)C_2 = 4\lambda C_1 + 4$$

$$C_2 = \frac{4 + 4\lambda C_1}{12-3\lambda} \quad \text{——— (5)}$$

From eqn(4) & eqn(5)

$$(20-5\lambda)C_1 - 4\lambda\left(\frac{4+4\lambda C_1}{12-3\lambda}\right) - 5 = 0$$

$$(20-5\lambda)(12-3\lambda)C_1 - 4\lambda(4+4\lambda C_1) - 5(12-3\lambda) = 0$$

$$[240 - 60\lambda - 60\lambda + 15\lambda^2]C_1 - 16\lambda - 16\lambda^2 C_1 - 60 + 15\lambda = 0$$

$$[240 - 120\lambda + 15\lambda^2 - 16\lambda^2]C_1 - 60 - \lambda = 0$$

$$C_1 = \frac{\lambda + 60}{240 - 120\lambda - \lambda^2} \quad \text{——— (6)}$$

From eqn (4)

$$(20-5\lambda)C_1 - 4\lambda C_2 - 5 = 0$$

$$C_1 = \frac{5+4\lambda C_2}{20-5\lambda}$$

From eqn (5)

$$C_2 = \frac{4+4\lambda C_1}{12-3\lambda}$$

$$(12-3\lambda)C_2 = 4+4\lambda\left[\frac{5+4\lambda C_2}{20-5\lambda}\right]$$

$$(12-3\lambda)(20-5\lambda)C_2 = 4(20-5\lambda) + 4\lambda(5+4\lambda C_2)$$

$$(240-60\lambda-60\lambda+15\lambda^2)C_2$$
$$= 80-20\lambda+20\lambda+16\lambda^2 C_2$$
$$= [240-120\lambda-\lambda^2]C_2 = 80$$

$$\Rightarrow C_2 = \frac{80}{240-120\lambda-\lambda^2} \quad \text{--- (7)}$$

From eqn (1), (6), (7)

$$g(x) = x + \lambda x C_1 + \lambda x^2 C_2$$

$$= x + \lambda x\left[\frac{\lambda+60}{240-120\lambda-\lambda^2}\right] + \lambda x^2\left[\frac{80}{240-120\lambda-\lambda^2}\right]$$

$$g(x) = \frac{x(240-120\lambda-\lambda^2)+\lambda^2 x+60\lambda x+80\lambda x^2}{240-120\lambda-\lambda^2}$$

$$g(x) = \frac{240x-120\lambda x-\lambda^2 x+\lambda^2 x+60\lambda x+80\lambda x^2}{240-120\lambda-\lambda^2}$$

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$$g(x) = \frac{240x - 120\lambda x + 60\lambda x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$$

$$g(x) = \frac{240x - 60\lambda x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$$

Q Solve $g(x) = (1+x^2) + \int_{-1}^{1} (xt + x^2t^2)\, g(t)\,dt$

Ans. $g(x) = 1 + x^2 + x\int_{-1}^{1} t\, g(t)\,dt + x^2\int_{-1}^{1} t^2 g(t)\,dt$

$g(x) = 1 + x^2 + xC_1 + x^2 C_2$ —— (1)

where $C_1 = \int_{-1}^{1} t\, g(t)\,dt$ —— (2)

$C_2 = \int_{-1}^{1} t^2\, g(t)\,dt$ —— (3)

From eqⁿ (1) & (2) then we get

$$C_1 = \int_{-1}^{1} t\,[1 + t^2 + tC_1 + t^2C_2]\,dt$$

$$C_1 = \int_{-1}^{1} t\,dt + \int_{-1}^{1} t^3\,dt + \int_{-1}^{1} t^2 C_1\,dt + C_2\int_{-1}^{1} t^3\,dt$$

$$= 0 + 0 + 2C_1\int_{0}^{1} t^2\,dt + 0$$

$$\left[\because \int_{-1}^{1} f(t)\,dt = \begin{cases} 0, & f \text{ is odd} \\ 2\int_0^1 f(t)\,dt, & f \text{ is even} \end{cases}\right]$$

$$C_1 = \frac{2C_1}{3}$$

$\Rightarrow 3C_1 - 2C_1 = 0 \Rightarrow \boxed{C_1 = 0}$ —— (4)

from eqn(3), (1) then we get

$$C_2 = \int_{-1}^{1} t^2\left[1+t^2+tC_1+t^2C_2\right]dt$$

$$= \int_{-1}^{1} t^2\,dt + \int_{-1}^{1} t^4\,dt + C_1\int_{-1}^{1} t^3\,dt + C_2\int_{-1}^{1} t^4\,dt$$

$$= 2\int_0^1 t^2\,dt + 2\int_0^1 t^4\,dt + 0 + 2C_2\int_0^1 t^4\,dt$$

$$C_2 = 2\left[\frac{t^3}{3}\right]_0^1 + 2\left[\frac{t^5}{5}\right]_0^1 + 2C_2\left[\frac{t^5}{5}\right]_0^1$$

$$C_2 = \frac{2}{3} + \frac{2}{5} + \frac{2}{5}C_2$$

$$15C_2 = 10 + 6 + 6C_2$$

$$9C_2 = 16$$

$$C_2 = \frac{16}{9} \quad \text{———} \quad (5)$$

From eqn (1), (4) & (5), we have

$$g(x) = 1 + x^2 + x\cdot 0 + x^2\cdot\frac{16}{9}$$

$$= 1 + x^2 + \frac{16}{9}x^2$$

$$\boxed{g(x) = 1 + \frac{25}{9}x^2}$$

9. Solve the homo. fredholm IE of 2nd kind

$$g(x) = \lambda\int_0^{2\pi} \sin(x+t)\,g(t)\,dt$$

Ex.- $g(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\, g(t)\, dt$

possesses no sol^n for $f(x) = x$ but that it possesses infinitely many sol^n, when $f(x) = 1$

Ans. The given I.E. can be written as

$$g(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, g(t)\, dt$$

$$g(x) = f(x) + \frac{1}{\pi}\left[\sin x \int_0^{2\pi} \cos t\, g(t)\, dt + \cos x \int_0^{2\pi} \sin t\, g(t)\, dt\right.$$

$$g(x) = f(x) + \frac{c_1}{\pi}\sin x + \frac{c_2}{\pi}\cos x \quad \text{---} (1)$$

where $c_1 = \int_0^{2\pi} \cos t \cdot g(t)\, dt$ --- (2)

& $c_2 = \int_0^{2\pi} \sin t\, g(t)\, dt$ --- (3)

Case - I - if $f(x) = x$

then (i) becomes

$$g(x) = x + \frac{c_1}{\pi}\sin x + \frac{c_2}{\pi}\cos x \quad \text{---} (4)$$

by eq^n (2) & (4)

$$c_1 = \int_0^{2\pi} \cos t \left[t + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right] dt$$

$$= \int_0^{2\pi} \left[t\cos t + \frac{c_1}{\pi}\sin t \cos t + \frac{c_2}{\pi}\cos^2 t\right] dt$$

$$= \int_0^{2\pi} t\cos t\, dt + \frac{c_1}{2\pi}\int_0^{2\pi} \sin 2t\, dt$$

$$+ \frac{c_2}{2\pi}\int_0^{2\pi} (1+\cos 2t)\, dt$$

$$\left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \Big[(t)\sin t + 1(\cos t)\Big]_0^{2\pi}$$

$$+ \frac{c_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{c_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\Rightarrow \quad c_1 = \frac{c_2}{2\pi}\cdot 2\pi \quad \Rightarrow \quad \boxed{c_1 = c_2} \quad \text{——— (5)}$$

Eqn (3) & (4)

$$c_2 = \int_0^{2\pi} \sin t \left[t + \frac{c_1}{\pi}\sin t + \frac{c_2}{\pi}\cos t\right] dt$$

$$= \int_0^{2\pi} t\sin t\, dt + \frac{c_1}{\pi}\int_0^{2\pi} \sin^2 t\, dt$$

$$+ \frac{c_2}{\pi}\int_0^{2\pi} \sin t\cos t\, dt$$

$$c_2 = \Big[t(-\cos t) - (1)(-\sin t)\Big]_0^{2\pi}$$

$$+ \frac{c_1}{2\pi}\int_0^{2\pi} (1-\cos 2t)\, dt + \frac{c_2}{2\pi}\Big(-\cos 2t\Big)_0^{2\pi}$$

$$= -2\pi + \frac{c_1}{2\pi}[2\pi - 0] + \frac{c_2}{2\pi}[-1+1]$$

$$c_2 = -2\pi + c_1$$

$$\Rightarrow \quad \boxed{c_1 - c_2 = 2\pi} \quad \text{——— (6)}$$

by eq^n(5) & (6)

$0 = 2\pi$

which is not possible so no solution which is absurd condition.
There is no sol^n

OR

$$C_1 - C_2 = 0$$
$$C_1 - C_2 = 2\pi$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\pi \end{bmatrix}$$

$$[A : b] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 2\pi \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$[A : b] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 2\pi \end{array}\right]$$

$\rho(A:b) = 2$ & $\rho(A) = 1$

$\therefore \rho(A:b) \neq \rho(A)$

$\therefore$ System is inconsistent & hence there is no sol^n.

Case-II If $f(x) = 1$ then (1) becomes

$$g(x) = 1 + \frac{C_1}{\pi}\sin x + \frac{C_2}{\pi}\cos x \quad \text{— (7)}$$

then by eq^n (3) & (7)

$$C_2 = \int_0^{2\pi} \sin t \left[1 + \frac{C_1}{\pi}\sin t + \frac{C_2}{\pi}\cos t\right] dt$$

$$= [-\cos t]_0^{2\pi} + \frac{C_1}{\pi}\int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \sin 2t\, dt$$

$$= \frac{C_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi}$$

$$\Rightarrow \quad C_2 = \frac{C_1 \cdot 2\pi}{2\pi}$$

$$\Rightarrow \quad \boxed{C_2 = C_1}$$

From eqn (2) & (7)

$$\boxed{C_1 = C_2 = C} \quad (\text{say})$$

then by eq. (7)

$$g(x) = 1 + \frac{C}{\pi}(\sin x + \cos x)$$

where c is arbitory constant

Given I.E. has infinite no. of soln for $f(x) = 1$

Q. Solve $g(x) = (1+x)^2 + \int_{-1}^{1} (xt + x^2t^2)\, g(t)\,dt$

$$\boxed{g(x) = 1 + 6x + \frac{25}{9}x^2}$$

Q. Solve the eqn $g(x) = 1 + \lambda \int_0^{\pi/2} \cos(x-t)\, g(t)\,dt$

find eigen value.

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☆ Solution of IE of II^{nd} kind by successive approximation -

~~In~~ Iterated Kernels

i) Consider the fredholm IE of II^{nd} kind is

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

then the iterated kernel $K_n(x,t)$ is defined as

$$K_1(x,t) = K(x,t)$$

and

$$K_n(x,t) = \int_a^b K(x,z)\, K_{n-1}(z,t)\, dz, \quad n = 2,3 \ldots$$

or

$$K_n(x,t) = \int_a^b K_{n-1}(x,z)\, K(z,t)\, dz \quad n = 2,3,\ldots$$

ii) Consider the volterra IE of 2^{nd} kind is

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)\, dt$$

then the iterated kernel $K_n(x,t)$ is defined as

$$K_1(x,t) = K(x,t)$$

&

$$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz, \quad n = 2,3 \ldots$$

or

$$K_n(x,t) = \int_t^x K_{n-1}(x,z)\, K(z,t)\, dz, \quad n = 2,3 \ldots$$

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✲ Resolvent kernel or Reciprocal kernel :-

a) Consider the 2nd kind fredhalm IE

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt \quad \text{--- (1)}$$

Soln of eq. (1) is given by

$$\boxed{g(x) = f(x) + \lambda \int_a^b R(x,t,\lambda)\, f(t)\, dt}$$

then $R(x,t,\lambda)$ is known as Resolvent kernel & given by

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

A G.P. eg. - $a + a^2 + a^3 + \dots$ is summable if series is convergent. & series is cong. if common ratio < 1

$$S_\infty = \frac{a}{1-r}$$

---

b) Consider the 2nd kind volterra IE

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)\, dt \quad \text{--- (2)}$$

soln of eqn (2) is given by

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$$g(x) = f(x) + \lambda \int_a^x R(x,t,\lambda)\, f(t)\, dt$$

then $R(x,t,\lambda)$ is known as resolvent kernel & given by

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

Q Find the iterated kernels for the following kernel

i) $K(x,t) = \sin(x-2t)$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

ii) $K(x,t) = x-t$, $a=0$, $b=1$

(i) Ans. Here $K(x,t) = \sin(x-2t)$

$\because$ $0 \le x, t \le 2\pi$ $\Rightarrow$ $a=0$, $b=2\pi$

Now let $K_1(x,t) = K(x,t)$
$= \sin(x-2t)$

& we know that iterated kernel

$$K_n(x,t) = \int_a^b K(x,z)\, K_{n-1}(z,t)\, dz \quad , n=2,3-$$

$$K_2(x,t) = \int_0^{2\pi} K(x,z)\, K_1(z,t)\, dz$$

$$= \frac{1}{2}\int_0^{2\pi} 2\sin(x-2z)\sin(z-2t)\, dz$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x-2z-z+2t) - \cos(x-2z+z-2t)]\, dz$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x+2t-3z) - \cos(x-2t-z)]\,dz$$

$$= \frac{1}{2}\left[\frac{\sin(x+2t-3z)}{-3} + \frac{\sin(x-2t-z)}{+1}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{\sin(x+2t-6\pi)}{-3} + \sin(x-2t-2\pi) + \frac{\sin(x+2t)}{3} - \sin(x-2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin[6\pi-(x+2t)] - \sin[2\pi-(x-2t)] + \frac{1}{3}\sin(x+2t) - \sin(x-2t)\right]$$

$$= \frac{1}{2}\left[-\sin(x+2t)\cdot\frac{1}{3} + \sin(x-2t) + \frac{1}{3}\sin(x+2t) - \sin(x-2t)\right]$$

$$= 0$$

$$\therefore \quad K_n(x,t) = 0$$

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Ex. Construct the iterated kernel of
$K(x,t) = e^{x+t}$, $a=0$, $b=1$

Ans- For the iterated kernel
$$K_1(x,t) = K(x,t) = e^{x+t}$$
&
$$K_n(x,t) = \int_a^b K(x,z)\, K_{n-1}(z,t)\, dz , \quad n=2,3,4$$

$$K_2(x,t) = \int_0^1 K(x,z)\, K_1(z,t)\, dz$$
$$= \int_0^1 e^{x+z} e^{z+t}\, dz = e^{x+t}\int_0^1 e^{2z}\, dz$$
$$= e^{x+t}\left[\frac{e^{2z}}{2}\right]_0^1 = \frac{e^{x+t}}{2}\left[e^2 - e^0\right]$$
$$= \frac{e^{x+t}}{2}\left[e^2 - 1\right]$$

$$K_3(x,t) = \int_0^1 K(x,z)\, K_2(z,t)\, dz$$
$$= \int_0^1 e^{x+z}\left[\frac{e^{z+t}(e^2-1)}{2}\right] dz$$
$$= \frac{e^{x+t}}{2}(e^2-1)\int_0^1 e^{2z}\, dz$$
$$= \frac{e^{x+t}}{2}(e^2-1)\left[\frac{e^2-1}{2}\right] = e^{x+t}\left[\frac{e^2-1}{2}\right]^2$$

$$K_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1} \quad m=1,2,3\ldots$$

We know that resolvent kernel
$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} e^{x+t} \left(\frac{e^2-1}{2}\right)^{m-1}$$

$$= e^{x+t} \sum_{m=1}^{\infty} \left[\lambda^{m-1} \left(\frac{e^2-1}{2}\right)^{m-1}\right]$$

$$= e^{x+t} \sum_{m=1}^{\infty} \left[\lambda \left(\frac{e^2-1}{2}\right)\right]^{m-1}$$

$$= e^{x+t} \frac{1}{1 - \frac{\lambda(e^2-1)}{2}}$$

provides $\left|\lambda\left(\frac{e^2-1}{2}\right)\right| < 1$

$$\Rightarrow |\lambda| < \frac{2}{e^2-1}$$

$$\boxed{R(x,t,\lambda) = \frac{2 \cdot e^{x+t}}{2-\lambda(e^2-1)}}$$

Q. Find the resolvent kernel for the following kernel

i) $K(x,t) = (1+x)(1-t)$; $a=-1$, $b=0$

ii) $K(x,t) = \sin x \cos t$; $a=0$, $b=\pi/2$

Q. Solve the following integral eqn

i) $g(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, g(t)\, dt$

ii) $g(x) = \left(e^x - \frac{e}{2} + \frac{1}{2}\right) + \frac{1}{2}\int_0^1 g(t)\, dt$

(i) Consider the FIE of 2nd kind

$$g(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, g(t)\, dt$$

Comparing above IE with standard form of FIE of 2nd kind, we get

$$f(x) = \frac{5x}{6}, \quad \lambda = \frac{1}{2} \quad \& \quad K(x,t) = xt$$

For the iterated kernel

$$K_1(x,t) = K(x,t) = xt$$

$$K_n(x,t) = \int_0^1 K(x,z)\, K_{n-1}(z,t)\, dz \quad , n = 2,3 \ldots$$

Now $K_2(x,t) = \int_0^1 K(x,z)\, K_1(z,t)\, dz$

$$= \int_0^1 xz \cdot zt\, dz = xt\int_0^1 z^2\, dz$$

$$K_2(x,t) = \frac{1}{3} \cdot xt$$

$$K_3(x,t) = \int_0^1 K(x,z)\, K_2(z,t)\, dz$$

$$= \int_0^1 xz \cdot \frac{1}{3} zt\, dz = \frac{xt}{3}\int_0^1 z^2 dz$$

$$K_3(x,t) = \frac{xt}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2 xt$$

$$K_m(x,t) = \left(\frac{1}{3}\right)^{m-1} xt$$

Therefore resolvent kernel

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \cancel{\sum_{m=1}^{\infty} \lambda'} \quad \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} \left(\frac{1}{3}\right)^{m-1} xt$$

$$= \sum_{m=1}^{\infty} \left(\frac{1}{2.3}\right)^{m-1} xt$$

$$= xt \left[1 + \frac{1}{2\cdot 3} + \left(\frac{1}{2.3}\right)^2 + \cdots \right]$$

$$= xt \left[\frac{1}{1 - 1/2\cdot 3} \right]$$

$$R(x,t,\lambda) = \frac{6}{5} xt$$

$\therefore$ Soln of given IE eqn

$$g(x) = f(x) + \lambda \int_0^1 R(x,t,\lambda) f(t)\, dt$$

$$= \frac{5x}{6} + \frac{1}{2} \int_0^1 \frac{6}{5} xt \cdot \frac{5}{6} \cancel{x}t \, dt$$

$$= \frac{5x}{6} + \frac{1}{2} x \int_0^1 t^2 \, dt$$

$$= \frac{5x}{6} + \frac{x}{6} = \frac{6x}{6}$$

$$\Rightarrow \boxed{g(x) = x}$$

ii) Consider the FIE of 2nd kind

$g(x) = f(x) + \lambda \int_0^1 K(x,t)\, g(t)\, dt$ ——— (1)

on comparing (1) by given IE

$f(x) = e^x - \frac{e}{2} + \frac{1}{2}$, $\lambda = \frac{1}{2}$, $K(x,t) = 1$

For the iterated kernel

$K_1(x,t) = K(x,t) = 1$

$K_n(x,t) = \int_0^1 K(x,z)\, K_{n-1}(z,t)\, dz$, $n = 2, 3, ---$

Now $K_2(x,t) = \int_0^1 K(x,z)\, K_1(z,t)\, dz$

$= \int_0^1 dz = [z]_0^1$

$K_2(x,t) = 1$

$K_3(x,t) = \int_0^1 K(x,z)\, K_2(z,t)\, dz = 1$

$K_m(x,t) = 1$

therefore resolvent kernel

$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$

$= \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} 1$

$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots = \frac{1}{1-\frac{1}{2}}$

$= 2$

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∴ Sol^n of the given integral eq^n

$$g(x) = f(x) + \lambda \int_0^1 R(x,t,\lambda)\, f(t)\, dt$$

$$= e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2} \int_0^1 2\left(e^t - \frac{e}{2} + \frac{1}{2}\right) dt$$

$$= e^x - \frac{e}{2} + 1/2 + 1/2 \cdot 2 \left[e^t - \frac{e}{2} t + \frac{1}{2} t \right]_0^1$$

$$= e^x - \frac{e}{2} + \cancel{3}1/2 + \left[e^1 - \frac{e}{2} + \frac{1}{2} - e^0 \right]$$

$$= e^x - \frac{e}{2} + \frac{1}{2} - \frac{e + e}{2} - \frac{1}{2} + 1$$

$$= e^x - \frac{e}{2} + 1/2 + \frac{e}{2} - 1/2 \qquad = e^x$$

$$\Rightarrow \boxed{g(x) = e^x}$$

Q Find the resolvent kernel of the following voltera kernel

i) $K(x,t) = 1$

Sol^n For the iterated kernel

$$K(x,t) = K_1(x,t) = 1$$

&

$$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz \quad , n = 2,3 \ldots$$

Now $K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$

$$= \int_t^x 1 \cdot 1\, dz = (x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= \int_t^x 1.(z-t)\, dz = \left[\frac{(z-t)^2}{2}\right]_t^x$$

$$K_3(x,t) = \frac{(x-t)^2}{2!}$$

$$K_m(x,t) = \frac{(x-t)^{m-1}}{(m-1)!}$$

Resolvent kernel is given

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(x-t)^{m-1}}{(m-1)!} = \sum_{m=1}^{\infty} \frac{[\lambda(x-t)]^{m-1}}{(m-1)!}$$

$$= \left[1 + \lambda(x-t) + \frac{\lambda^2 (x-t)^2}{2!} + \cdots\right]$$

$$R(x,t,\lambda) = e^{\lambda(x-t)}$$

ii) $K(x,t) = e^{x-t}$

Soln For the iterated kernel

$$K(x,t) = K_1(x,t) = e^{x-t}$$

&

$$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz \quad , \quad n = 2,3 \ldots$$

$$K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$$

$$= \int_t^x e^{x-z} e^{z-t}\, dz$$

$$= e^{x-t} \int_t^x dz = e^{x-t}(x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

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$$= \int_t^x e^{x-z} e^{z-t} (z-t)\, dz = e^{x-t} \int_t^x (z-t)\, dz$$

$$= e^{x-t} \left[\frac{(z-t)^2}{2}\right]_t^x$$

$$= e^{x-t} \frac{(x-t)^2}{2!}$$

$$\Rightarrow K_m(x,t) = e^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}$$

Resolvent kernel is given

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} e^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}$$

$$= e^{x-t} \sum_{m=1}^{\infty} \frac{[\lambda(x-t)]^{m-1}}{(m-1)!}$$

$$= e^{x-t} \left[1 + \lambda(x-t) + \frac{\lambda^2(x-t)^2}{2!} + \cdots\right]$$

$$= e^{x-t} e^{\lambda(x-t)}$$

iii) $K(x,t) = a^{x-t}$

For iterated kernel

$$K(x,t) = K_1(x,t) = a^{x-t}$$

$$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz$$

$$K_2(x,t) = \int_t^x a^{x-z} a^{z-t}\, dz = a^{x-t} \int_t^x dz$$

$$K_2(x,t) = a^{x-t}(x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= \int_t^x a^{x-z}\, a^{z-t}\,(z-t)\, dz$$

$$= a^{x-t} \int_t^x (z-t)\, dz$$

$$= a^{x-t} \left[\frac{(z-t)^2}{2} \right]_t^x$$

$$K_3(x,t) = a^{x-t} \frac{(x-t)^2}{2}$$

$$\therefore \quad K_m(x,t) = a^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}$$

Resolvent kernel is

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} a^{x-t} \frac{(x-t)^{m-1}}{(m-1)!}$$

$$= a^{x-t} \sum_{m=1}^{\infty} \frac{[\lambda(x-t)]^{m-1}}{(m-1)!}$$

$$= a^{x-t} \left[1 + \lambda(x-t) + \frac{\lambda^2 (x-t)^2}{2!} + \cdots \right]$$

$$R(x,t,\lambda) = a^{x-t} e^{\lambda(x-t)}$$

iv) $K(x,t) = \dfrac{2+\cos x}{2+\cos t}$

Soln For the iterated kernel

$$K(x,t) = K_1(x,t) = \frac{2+\cos x}{2+\cos t}$$

&

$$K_n(x,t) = \int_t^x K(x,z)\, K_{n-1}(z,t)\, dz$$

$$K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz$$

$$= \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x dz$$

$$K_2(x,t) = \frac{2+\cos x}{2+\cos t}\,(x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}\,(z-t)\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x (z-t)\, dz$$

$$K_3(x,t) = \frac{2+\cos x}{2+\cos t} \left[\frac{(z-t)^2}{2}\right]_t^x$$

$$= \frac{2+\cos x}{2+\cos t} \frac{(x-t)^2}{2!}$$

$$\therefore K_m(x,t) = \frac{2+\cos x}{2+\cos t} \frac{(x-t)^{m-1}}{(m-1)!}$$

Resolvent kernel is

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{2+\cos x}{2+\cos t} \frac{(x-t)^{m-1}}{(m-1)!}$$

$$= \frac{2+\cos x}{2+\cos t} \sum_{m=1}^{\infty} \frac{[\lambda(x-t)]^{m-1}}{(m-1)!}$$

$$= \frac{2+\cos x}{2+\cos t} \left[1 + \lambda(x-t) + \frac{\lambda^2 (x-t)^2}{2!} + \cdots \right.$$

$$R(x,t,\lambda) = \frac{2+\cos x}{2+\cos t} e^{\lambda(x-t)}$$