Differential Equations

UNIT- V

By

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Numerical Solutions of O.D.E.

Initial Value Problem (IVP) $\rightarrow$ If all the conditions that are required to solve ordinary an D.E. are given at a single point, then the problem is said to be IVP.

Example :- $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

$y''' = f(x, y, y', y'')$ with $y(1) = 0$
$y'(1) = 1$
$y''(1) = 4$

$y'' = f(x, y, y')$ with $y(0) = 1$
$y'(0) = 2$

Boundary Value Problem (B.V.P.) $\rightarrow$ If all the imposed conditions that are required to solve ordinary differential eq^n. are given at more than one points, then the problem is called BVP.

Example - $y''' = f(x, y, y', y'')$

$y(0) = 1$
$y'(1) = 2$
$y''(2) = 3$

Step length if large, No. of Error ↑
if small, no of approximation ↑

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Euler's Method

Consider I.V.P. $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

By Euler's method, n^{th} approximation of y is given as

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$
$n = 1, 2, 3, \ldots$

where h is step size or length

Shortcomings of the method:-

1. If the step length or size is small, the method is too slow (large no. of iterations).
2. If the step length is large, then the method is inaccurate (error is large).

Use Euler's method to determine an approximate value of y at $x = 0.2$ from the IVP.

$$\frac{dy}{dx} = 1 - x + 4y, \quad y(0) = 1$$ taking the step size $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$
$n = 1, 2, 3, \ldots$

Ist approximation $y(x_1) = y_1 = y_0 + (0.1) f(x_0, y_0)$
$= 1 + 0.1 (1 - 0 + 4 \times 1)$

$= 1.5$

II approximation. Now: $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.5 + 0.1[1 - x_1 + 4y_1]$$
$$= 1.5 + 0.1[1 - 0.1 + 4 \times 1.5]$$

$$= 2.19$$

$\therefore y(0.2) = 2.19$

25|2|20 Eulers Modified Method

Consider the IVP.

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

$$y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1})$$
$n = 1$ [redacted]

$$y_n^{(m)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{m-1})\right]$$

$m = 1, 2, 3, \ldots$

Using Eulers modified Method, obtain a soln of the eqn

$$\frac{dy}{dx} = x + |\sqrt{y}|$$

With the initial condition $y = 1$ at $x = 0$ for the range $0 \le x \le 0.6$ in steps of 0.2

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correct upto 4 places of decimals.

$x_0 = 0, \; y_0 = 1, \; h = 0.2$

$\cancel{y_1}$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

At $x = 0.2$

$(y_1)_I = y_0 + h f(x_0, y_0)$

$= 1 + 0.2 [0 + |\sqrt{1}|] = 1.2$

$(y_1)_{II} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, (y_1)_I)]$

$= 1 + \frac{0.2}{2} [0 + |\sqrt{1}| + 0.2 + |\sqrt{1.2}|]$

$= 1 + 0.1 [1 + 0.2 + 1.0954] = 1.2295$

$(y_1)_{III} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, (y_1)_{II})]$

$= 1 + \frac{0.2}{2} [0 + |\sqrt{1}| + 0.2 + (\sqrt{1.2295})]$

$= 1.2309$

$(y_1)_{IV} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, (y_1)_{III})]$

$= 1.2309$

$(y_1)_{III} = (y_1)_{IV}$

Hence $y_1 = y(x_1) = y(0.2) = 1.2309$

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Now $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

Therefore at $x = 0.4$.

$f(x_1, y_1) = 0.2 + \sqrt{1.2309}$
$= 0.2 + 1.1094$
$= 1.3094$

$(y_2)_I = y_1 + hf(x_1, y_1)$

$= 1.2309 + 0.2\left[0.2 + |\sqrt{1.2309}|\right]$
$= 1.4927$

$(y_2)_{II} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, (y_2)_I)\right]$

$= 1.5240$

$(y_2)_{III} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, (y_2)_{II})\right]$

$= 1.5253$

$(y_2)_{IV} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, (y_2)_{III})\right]$

$= 1.5253$

$(y_2)_{III} = (y_2)_{IV}$. Hence $y_2 = 1.5253$

Now, $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$(y_3)_I = y_2 + hf(x_2, y_2)$

$= 1.5253 + 0.2\left[0.4 + \sqrt{1.5253}\right]$

$= 1.5253 + 0.2\left[0.4 + 1.2350\right] = 1.8523$

1.63
0.3

$(y_3)_{II} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, (y_3)_I)\right]$

$= 1.5253 + \frac{0.2}{2}\left[1.6350 + 0.6 + \sqrt{1.8523}\right]$

$= 1.5253 + 0.1\left[3.5959\right] = 1.8849$

$(y_3)_{III} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, (y_3)_{II})\right]$

$= 1.5253 + 0.1\left[1.6350 + 0.6 + 1.3729\right]$

$= 1.88609 \approx 1.8861$

$(y_3)_{IV} = 1.8861$ Hence $y_3 = 1.8661$

Picard's Method of successive approximations:-

(Picard's Iterative method)

Consider I.V.P.,

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0$$

By Picard's method, n^{th} approximation

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad n = 1, 2, 3, 4, \ldots$$

Solve the D.E. :-

$\frac{dy}{dx} = x + y^2$, $y(0) = 0$ by Picard's method upto third ~~order~~ approximation of solution of the DE.

$\Rightarrow$ Consider IVP.

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0$$

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Given problem is

$f(x,y) = x + y^2$, $x_0 = 0$, $y_0 = 0$.

$\because$ We know that n^{th} approximation of Picard's method is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{---} (1)$$

$$n = 1, 2, 3, \ldots$$

I^{st} approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 0 + \int_0^x f(x, 0)\,dx$$

$$= \int_0^x x\,dx = \frac{x^2}{2}\Big|_0^x = \frac{x^2}{2}$$

2^{nd} approximation,

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx$$

$$= 0 + \int_0^x (x + y_1^2)\,dx$$

$$= \int_0^x \left(x + \frac{x^4}{4}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

3^{rd} Approximation $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$

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$$y_3 = 0 + \int_0^x (x + y_2^2)\,dx$$

$$= \int_0^x \left\{x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right\} dx$$

$$= \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{2x^7}{40}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^{11}}{4400} + \frac{x^8}{160}$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

Picards method for Simultaneous D. Equations :

Consider the simultaneous equations of first order

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y,z) \\ \frac{dz}{dx} &= g(x,y,z) \end{aligned}\right\} \quad (1)$$

with $y(x_0) = y_0$ and $z(x_0) = z_0$

By Picard's method, Ist approximation of the solⁿ is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx$$

and $z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)\,dx$

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II approximation is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\,dx$$

and $$z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)\,dx$$

n^{th} approximation is

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx$$

and $$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx$$

where $n = 1, 2, 3, \ldots$

Use Picard's method to solve $\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} + x^4 y$

$y = 5$ and $\frac{dy}{dx} = 1$ when $x = 0$.

Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$.

Then the given D.E. can be written as:

$$\frac{dz}{dx} = x^2 z + x^4 y$$

$\therefore$ Given initial value problem can be written as

$$\frac{dy}{dx} = z = f(x, y, z)$$

$$\frac{dz}{dx} = x^2 z + x^4 y = g(x, y, z)$$ with $y(0) = 5$ and $z(0) = 1$.

I approximation of the solution is

[Atleast 3 iterations]

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$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx$$

$$y_1 = 5 + \int_0^x dx = 5 + x$$

and $z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)\, dx$

$$z_1 = 1 + \int_0^x (x^2 + 5x^4)\, dx$$

$$z_1 = 1 + \frac{x^3}{3} + x^5$$

2^{nd} approximation of the solution is:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx$$

$$= 5 + \int_0^x \left(1 + \frac{x^3}{3} + x^5\right) dx = 5 + x + \frac{x^4}{12} + \frac{x^6}{6}$$

z_1

and $z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)\, dx$

$$z_2 = 1 + \int_0^x \left[x^2\left(1 + \frac{x^3}{3} + x^5\right) + x^4(5 + x)\right] dx$$

$$= 1 + \int_0^x \left(x^2 + \frac{x^5}{3} + x^7 + 5x^4 + x^5\right) dx$$

$$= 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^8}{8} + x^5 + \frac{x^6}{6}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8}$$

3^{rd} approximation of the solution is:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\,dx$$

$$= 5 + \int_{x_0}^{x} \left(1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8}\right) dx$$

$$= 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2}{63}x^7 + \frac{x^9}{72}$$

We need to find Z_3.

$$Z_3 = Z_0 + \int_{x_0}^{x} g(x, y_2, z_2)\,dx$$

$$= 1 + \int_{x_0}^{x} \left[x^2\left(1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8}\right) + x^4\left(5 + x + \frac{x^4}{12} + \frac{x^6}{6}\right)\right] dx$$

$$= 1 + \int_{x_0}^{x} \left(x^2 + \frac{x^5}{3} + x^7 + \frac{2}{9}x^8 + \frac{x^{10}}{8} + 5x^4 + x^5 + \frac{x^8}{12} + \frac{x^{10}}{6}\right) dx$$

$$= 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^8}{8} + \frac{2}{81}x^9 + \frac{x^{11}}{88} + x^5 + \frac{x^6}{6} + \frac{x^9}{108} + \frac{x^{11}}{66}$$

$$= 1 + \frac{x^3}{3} + x^5 + \frac{2}{9}x^6 + \frac{x^8}{8} + \frac{7x^9}{324} + \frac{7x^{11}}{264}$$