Calculus

Paper-II

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MATHS CALCULUS

ASYMPTOTES : Asymptote is an equation of a line which has finite distance from origin and distance of touch point from origin tends to infinity.

To find the asymptote of the curve $y = f(x)$

Let curve eq. be $y = f(x)$ ——— ①

Therefore, eq of tangent at (x, y) is

$$Y - y = m(X - x)$$

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y = \frac{dy}{dx} X + \left(y - x\frac{dy}{dx}\right) \quad \text{——— ②}$$

If we skip asymptotes parallel to y axis then $x \to \infty$, $\frac{dy}{dx}$ does tends to infinity.

As x extending to infinity, $x \to \infty$

$\frac{dy}{dx}$ and $y - x$ are finite.

let $\lim_{x \to \infty} \frac{dy}{dx} = m$

and $\lim_{x \to \infty} y - x\frac{dy}{dx} = c$

$$\Rightarrow \lim_{x \to \infty}\left(\frac{y}{x} - \frac{dy}{dx}\right) = \lim_{x \to \infty} \frac{c}{x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{y}{x} - \lim_{x \to \infty} \frac{dy}{dx} = 0$$

$$\Rightarrow \lim_{x \to \infty} \frac{y}{x} = m$$

Eq. ② becomes $Y = mx + c$

① Diff. Calculus NK Publication
② Integral Calculus
J.P.H Dr. MP Goswami

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Asymptote

Parallel Asymptote

x axis parallel
y axis parallel

If you are finding the parallel asymptotes to x axis then put the coefficients of highest power of $x = 0$. That obtained eq. is parallel asymptote to X axis.

If coefficient is constant then there is no parallel asymptote for X axis.

For finding the parallel asymptote to y axis then put the coefficient of highest power of $y = 0$. That obtained eq. is parallel asymptote to y axis. If coefficient is constant then there is no parallel asymptote for Y axis.

Oblique Asymptotes

$y = mx + c$

find

Let eq. of curve $y = f(x)$ of n degree.

I Put $x = 1$ and $y = m$ in curve eq. and then find

$\phi_n(m) =$

$\phi_{n-1}(m) =$

$\vdots$

$\phi_1(m) =$

II For finding value of m put

$$\phi_n(m) = 0 \quad \text{—— ①}$$

Here ① ~~degree~~ is n degree equation. Therefore it has **ATMOST** n real roots.

Case 1: if m are distinct then

$$c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)} \qquad m = m_1$$

$$\therefore \boxed{y = m_1 x + c}$$

where $\phi_n'(m)$ is $\frac{d}{dm}\phi_n(m)$

If a curve has n degree equation then the curve has ATMOST n asymptotes.
ATLEAST 0 asymptotes.
i.e. $0 \leq$ no. of asymptotes $\leq n$

Contd...

Oblique Asymptotes

$$y = mx + c$$

let eq of curve $y = f(x)$ of n degree.

I Put $x = 1$ and $y = m$ in curve eq & then find

$\phi_n(m) =$

$\phi_{n-1}(m) =$

$\vdots$

$\phi_1(m) =$

II For finding value of m, put $\phi_n(m) = 0$ —— (1)
Here (1) is n degree eq. Therefore it has atmost n real roots.

Case 1: if m are distinct then

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)} \qquad m = m_1$$

$$\therefore \boxed{y = m_1 x + c}$$

where $\phi_n'(m) = \frac{d}{dm}\phi_n(m)$

Case 2: If roots are repeated say $m_1 = m_2$
then for c,

$$\left.\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m)\right|_{m=m_1} = 0$$

Let $m_1 = m_2 = m_3$ then for c

$$\left.\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + c\,\phi_{n-2}'(m) + \phi_{n-3}(m)\right|_{m=m_1} = 0$$

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$\phi_n(m) = 0$ $\phi_n'(m) = 0$

$\phi_{n-1}(m) \neq 0$

$\therefore$ Asymptote will not exist.

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Ex Find the asymptote of the curve

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 = 1$$

$\because$ Coefficients of highest power of x and y are constants. Therefore, there is no parallel asymptotes to Axis.

Now for oblique asymptote :-

Put $x = 1$ and $y = m$ in given curve eq., we have

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3$$

$$\phi_2(m) = m - m^2$$

$$\phi_1(m) = 0$$

For finding value of m

put $\phi_3(m) = 0$

$$2m^3 + m^2 - 2m - 1 = 0$$

$$(2m+1)(m^2-1) = 0$$

$$(2m+1)(m-1)(m+1) = 0$$

$$\Rightarrow m = 1, -1, \frac{-1}{2}$$

Now,

$$\phi_3'(m) = 2 - 2m - 6m^2$$

for $m = -1$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(m - m^2)}{2 - 2m - 6m^2}\Bigg|_{m=-1}$$

$$c = -\left(\frac{-2}{-2}\right) = -1$$

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Therefore, asymptote for $m = -1$ is

$$y = -x - 1$$

$$\Rightarrow \boxed{x + y + 1 = 0}$$

For $m = 1$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)}\Bigg|_{m=1} = -\frac{(m - m^2)}{(2 - 2m - 6m^2)}\Bigg|_{m=1}$$

$$c = 0$$

$\therefore$ Asymptote for $m = 1$ is $\boxed{y = x}$

For $m = \frac{-1}{2}$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)}\Bigg|_{m=-1/2}$$

$$= -\frac{(m - m^2)}{(2 - 2m - 6m^2)}\Bigg|_{m=-1/2}$$

$$= -\left(\frac{-1}{2} - \frac{1}{4}\right) \div \left(2 + 1 - \frac{6}{4}\right)$$

$$c = -\left(\frac{-3}{4}\right) \div \frac{3}{2} = \frac{1}{2}$$

Therefore asymptote for $m = -1/2$ is

$$y = \frac{-1}{2}x + \frac{1}{2} \Rightarrow \boxed{2y + x - 1 = 0}$$

$$\Rightarrow \boxed{x + 2y - 1 = 0}$$

Thus, asymptote of given curve are

$x + y + 1 = 0$, $x = y$, $x + 2y - 1 = 0$

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Q Find asymptotes of the curve

$$y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$$

∵ coefficient of highest power of x and y are constant.
Therefore there is no parallel asymptote to axis.

→ Now for oblique asymptotes put $x = 1$ and $y = m$ is given curve eq., we have

$$\phi_3(m) = m^3 - m^2 - m + 1$$

$$\phi_2(m) = 1 - m^2$$

$$\phi_1(m) = 0$$

For finding the value of m, put $\phi_3(m) = 0$

$$m^3 - m^2 - m + 1 = 0$$

$$m^2(m-1) - 1(m-1) = 0$$

$$(m^2-1)(m-1) = 0 \quad \Rightarrow (m-1)(m+1)(m-1) = 0$$

$$\Rightarrow m = 1, 1, -1$$

For $m = 1$, $m = -1$

$$c = \left.\frac{\phi_2(m)}{\phi_3'(m)}\right|_{m=-1} = \left.\frac{1-m^2}{3m^2-2m-1}\right|_{m=-1}$$

$$c = 0$$

$$y = mx \Rightarrow y = -x$$

$$\Rightarrow x + y = 0$$

For $m = 1$, Repeated Roots

$$\left.\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m)\right|_{m=1} = 0$$

$$\left.\frac{c^2}{2}(6m-2) + c(-2m) + 0\right|_{m=1} = 0$$

$$2c^2 - 2c = 0 \Rightarrow c(c-1) = 0 \Rightarrow c = 1, 0$$

∴ Corresponding asymptotes for $m = 1$ are :-

$$y = mx + c \Rightarrow y = x + 0, \quad c = 0$$

$$\text{and } y = x + 1, \quad c = 1$$

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Thus asymptote of given curve are

$$\boxed{x+y=0, \quad x-y=0, \quad x-y+1=0}$$

Q Find the asymptotes of the curve

$$x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0$$

Since coefficients of highest power of x & y are constant. So, there is no parallel asymptotes to Axis

Put $x=1$ $\quad y=m$

$$\phi_3(m) = 1 - 5m + 8m^2 - 4m^3$$

$$\phi_2(m) = 1 - 3m + 2m^2$$

$$\phi_1(m) = 0$$

For values of m,

$$\phi_3(m) = 0$$

$$1 - 5m + 8m^2 - 4m^3 = 0$$

$$-4m^3 + 8m^2 - 5m + 1 = 0$$

$$m-1 \,\big)\, -4m^3 + 8m^2 - 5m + 1 \,\big(\, -4m^2 + 4m - 1$$

$$\underline{-4m^3 + 4m^2}$$

$$4m^2 - 5m + 1$$

$$\underline{\overset{-}{4m^2} \overset{+}{-} 4m}$$

$$-m+1$$

$$\Rightarrow -4m^3 + 8m^2 - 5m + 1 = 0$$

$$\Rightarrow (-4m^2 + 4m - 1)(m-1) = 0$$

$$\Rightarrow (2m-1)^2(m-1) = 0 \qquad \Rightarrow m = \tfrac{1}{2}, \tfrac{1}{2}, 0$$

For $m = \tfrac{1}{2}$,

$$\left.\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m)\right|_{m=1/2} = 0$$

$$\frac{c^2}{2}(16 - 24m) + c(-3 + 4m) + 0 = 0$$

$$\Rightarrow 2c^2 - 3c + 2c = 0 \qquad \Rightarrow 2c^2 - c = 0 \qquad \Rightarrow c = \frac{1}{2}, 0$$

asymptotes for $m = 1/2$ when $c = 1/2, 0$

$y = \frac{x}{2} + \frac{1}{2}$ $\qquad$ $y = \frac{x}{2}$

$\boxed{x - 2y + 1 = 0}$ $\qquad$ $\boxed{x - 2y = 0}$

For $m = 1$, $c = \frac{\phi_2(m)}{\phi_3'(m)} = \left.\frac{1 - 3m + 2m^2}{-5 + 16m - 12m^2}\right|_{m=1}$

$= \frac{1 - 3 + 2}{-5 + 16 - 12} = 0$

$y = x \Rightarrow \boxed{y - x = 0}$

So, Asymptotes are

$x - 2y + 1 = 0, \quad y - x = 0, \quad x - 2y = 0$

Ex- $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

$a^2y^2 - b^2x^2 - x^2y^2 = 0$

4 degree curve

$x^2 = a^2 \qquad b^2 = -y^2$

Intersection of the Curve & its Asymptotes.

Let $y = f(x)$ be a n degree curve, since we know that a line cut the curve of n degree at n points since in asymptotes, 2 cut pts. converted into a touch pt. at infinity (∞). Therefore, n-asymptotes cut n-2 points of the given curve. If curve has m asymptotes then total no. of intersection pts. $= m(n-2)$

Let curve of n degree whose asymptotes equation $P_1 = 0$

Eq. of curve $P_2 = 0$

then the eq. of the curve which passes through intersection pts. of curves and its asymptote is $P_1 + \lambda P_2 = 0$

By putting value of λ, we obtain the req. curve eq.

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Q Show that the asymptotes of the cubic curve

$$x^3 - 2y^3 + xy(2x-y) + y(x-y) + 1 = 0 \quad \text{—— (1)}$$

cut the curve again in 3 pts. which lie on the straight line $x - y + 1 = 0$

Since highest power of x and y has constant coefficients,
∴ there is no parallel asymptotes to axis.

For oblique asymptotes put $x = 1$, $y = m$ in curve eq.

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2$$

$$\phi_2(m) = m - m^2$$

$$\phi_1(m) = 0$$

For value of m, put $\phi_3(m) = 0$

$$2m^3 + m^2 - 2m - 1 = 0$$

$$(m-1)(2m^2 + 3m + 1) = 0$$

$$m = -1, 1, -1/2$$

For $m = -1$, $c = -\dfrac{\phi_2(m)}{\phi_3'(m)}\Bigg|_{m=-1} = \dfrac{-(m-m^2)}{-6m^2+2-2m} = \dfrac{2}{-2} = -1$

$y = -x - 1 \Rightarrow \boxed{x + y + 1 = 0}$

For $m = 1$ $\quad c = 0$

So, $y = x \Rightarrow \boxed{y - x = 0}$

For $m = -1/2$, $c = -\dfrac{(-1/2 - 1/4)}{-3/2 + 2 + 1} = \dfrac{3}{4} \times \dfrac{2}{3} = \dfrac{1}{2}$

$y = -\dfrac{x}{2} + \dfrac{1}{2} \Rightarrow \boxed{x + 2y - 1 = 0}$

Therefore total no. of Asymptotes $= 3$

Thus, total no. of Intersection pts. of asymptotes & curve $= m(n-2)$

$$= 3(3-2) = 3$$

∴ Asymptotes cut the given curve at 3 pts.

→ Now, combined eq. of Asymptote :-

$$(x+y+1)(x-y)(x+2y-1) = 0$$

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$\Rightarrow (x^2 - xy + xy - y^2 + x - y)\ (x + 2y - 1) = 0$

$\Rightarrow (x^2 - y^2 + x - y)\ (x + 2y - 1) = 0$

$\Rightarrow x^3 - xy^2 + x^2 - yx + 2yx^2 - 2y^3 + 2yx - 2y^2 - x^2 + y^2 - x + y = 0$

$\Rightarrow x^3 - 2y^3 + 2x^2y - xy^2 - y^2 + xy - x + y = 0$ —— ②

Now ① - ② we get

$1 + x - y = 0 \quad \Rightarrow \boxed{x - y + 1 = 0}$

which is required curve eq. which passes through intersection of curve.

Ex. Prove that asymptotes of the curve

$xy(x^2 - y^2) + 25y^2 + 9x^2 - 144 = 0$ —— ①

cut it again in 8 pts line on an ellipse whose eccentricity is 4/5.

Parallel asymptote $x = 0$, $y = 0$

Given curve: $x^3y - xy^3 + 25y^2 + 9x^2 - 144 = 0$

For oblique, $x = 1 \quad y = m$

$\phi_4(m) = m - m^3 \cancel{+25m}$

$\phi_3(m) = 0$

for value of m, put $\phi_4(m) = 0$

$m(1 - m^2) = 0$

$m(1 - m)(1 + m) = 0$

$\Rightarrow m = 0, 1, -1$

for $m = 1$ $\quad c = -\dfrac{\phi_3(m)}{\phi_4'(m)} = 0$

So, $y = x \quad \Rightarrow x - y = 0$

Similarly

for $m = -1$, $\quad c = 0$

$y = -x \quad \Rightarrow x + y = 0$

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Asymptotes of the given curve

$x=0 \qquad y=0 \qquad x-y=0 \qquad x+y=0$

No. of intersection pts. $= 4(4-2) = 8$

Asymptotes cut the given curve at 8 pts..

$\therefore$ Combined eq. of asymptote is

$$xy\,(x^2-y^2)=0 \qquad \text{———} \quad (2)$$

Now (1) $-$ (2),

$$25y^2+9x^2-144=0$$

$$\Rightarrow 9x^2+25y^2=144$$

$$\Rightarrow \frac{x^2}{(12/3)^2}+\frac{y^2}{(12/5)^2}=1$$

$$\Rightarrow \frac{x^2}{4^2}+\frac{y^2}{(12/5)^2}=1$$

$\because b^2=a^2(1-e^2)$

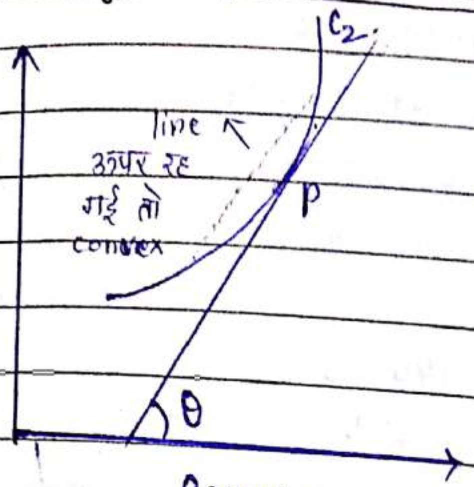
so, $e=\sqrt{1-\left(\frac{9}{25}\right)^2} = \frac{4}{5}$

Hence Proved.

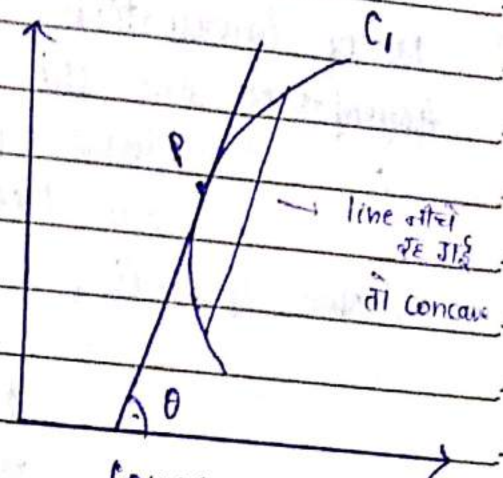
Chapter-4

Multiple Points & Curve Tracing.

Convexity and concavity of a curve



Convex



Concave

If curve below the tangent then the curve is said to be CONCAVE and if the ~~tang~~ curve above the tangent then the curve, is said to be CONVEX.

A curve $y = f(x)$ is said to be CONVEX wrt X axis, if

if $\dfrac{d^2y}{dx^2} > 0$

and the curve is convex wrt y axis,

if $x\dfrac{d^2x}{dy^2} > 0$

A curve $y = f(x)$ is said to be concave wrt X axis if

if $y^2\dfrac{d^2y}{dx^2} < 0$

and the curve is concave wrt y-axis

if $x\dfrac{d^2x}{dy^2} < 0$

A curve $y = f(x)$ is said to be convex wrt X axis in close interval $[a, b]$ if

$$\frac{d^2y}{dx^2} > 0 \qquad \forall [a,b]$$

III A curve $y = f(x)$ is said to be concave wrt X axis

$$\text{if} \quad \frac{d^2y}{dx^2} < 0 \qquad \forall [a,b]$$

POINT OF INFLEXION

A point P is said to be a point of inflexion if one side of the point of the curve is convex and in another side curve is concave if

one side $\frac{d^2y}{dx^2} > 0$ and $\frac{d^2y}{dx^2} < 0$ on another side.

i.e. a point P is said to be point of inflexion

if $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$ at P.

In general, a point P will be pt. of inflexion if

$$f''(x) = f'''(x) = f''''(x) = \dots = f^n(x) = 0$$

and

$$f^{(n+1)}(x) \neq 0, \text{ where } n \text{ is even no.}$$

Example- Find the point of inflexion of the curve $y = [\log x]^3$

$$\frac{dy}{dx} = 3(\log x)^2 \left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = 3\left[-\frac{1}{x^2}(\log x)^2 + \frac{2}{x^2}\log x\right]$$

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$$= \left[\frac{6\log x}{x^2} - \frac{3}{x^2}(\log x)^2\right]$$

$$= \frac{3}{x^2}\log x\left[2.\log e - \log x\right] \qquad \log e = 1$$

$$= \frac{3}{x^2}\log x\left[\log e^2 - \log x\right]$$

$$= \frac{3}{x^2}\log x\left(\log\frac{e^2}{x}\right)$$

For pt. of inflexion, $\frac{d^2y}{dx^2} = 0$

$$\frac{3}{x^2}\log x \log\left(\frac{e^2}{x}\right) = 0$$

So either $\log x = 0$ or $\log(e^2/x) = 0$

$\Rightarrow x = e^0$ $\qquad \frac{e^2}{x} = 1 \qquad \because e^0 = 1$

$\Rightarrow x = 1$

$\Rightarrow x = e^2$

For $x = 1$

$$y = (\log x)^3 = 0$$

$(1,0) \qquad (e^2, 8)$

$$\frac{d^3y}{dx^3} = \frac{-1}{x}\left(\frac{3}{x^2}\log x\right) + \left(2 - \log x\right)\left[\frac{-6}{x^3}\log x + \frac{3}{x^3}\right]$$

$$= \left(\frac{3}{x^3} - \frac{6\log x}{x^3}\right)\log\frac{e^2}{x} - \frac{3\log x}{x^3}$$

$$\therefore \left.\frac{d^3y}{dx^3}\right|_{(1,0)} = \frac{3}{1}\times 2 = 6 \neq 0$$

$$\therefore \left.\frac{d^3y}{dx^3}\right|_{(e^2,8)} \left(\frac{3}{e^6} - \frac{12}{e^6}\right)0 - 3\frac{\log e^2}{e^6} = 0 - 3\left(\frac{2}{e^6}\right)$$

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$$= -\frac{6}{e^6} \neq 0$$

Therefore pts $(1,0)$ & $(e^2,8)$ are pts of inflexion of the given curve.

Q Find the ranges of ~~th~~ values at x in which the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave & convex. Also, find the pts of inflexion.

$$\frac{dy}{dx} = 15x^4 - 120x^2 + 3$$

$$\frac{d^2y}{dx^2} = 60x^3 - 240x \qquad \text{for concave}$$

$$\frac{d^3y}{dx^3} = 180x^2 - 240$$

$$\frac{d^2y}{dx^2} < 0$$

$$\Rightarrow 60x(x^2-4) < 0$$

For point of inflexion $\frac{d^2y}{dx^2} = 0$

$$60x(x^2-4) = 0$$

$$\Rightarrow x = 0 \qquad x = \pm 2$$

At 0, $y = -20$

at $+2$, $y = -238$

at -2, $y = 198$

Now at point P, $\frac{d^3y}{dx^3} = -240 \neq 0$

Therefore, P are the pts of inflexion.

$$\frac{d^2y}{dx^2} = 60x(x^2-4)$$

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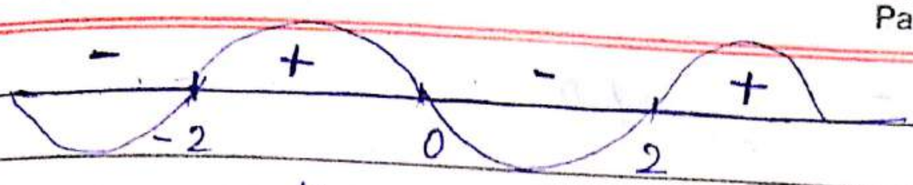


For convex, $\frac{d^2y}{dx^2} > 0$ $(-2,0) \cup (2,\infty)$

for concave, $\frac{d^2y}{dx^2} < 0$ $(-\infty,-2) \cup (0,2)$

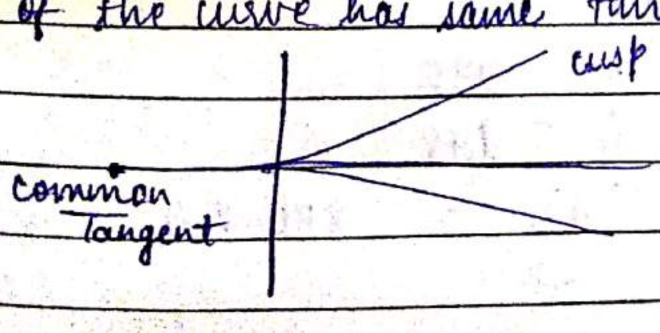
Multiple Point (Double Point)
A point of the curve is said to be multiple point if more than one branches of the curve passes through through the point.

* Double Point :- A point on the curve is said to be double points if two branches of the curve passes through that point.

SOME FUNDAMENTAL DEFINITION

1. Node : In double point on the curve is said to be NODE if there are 2 distinct tangent at 2 branches of the curve.

2. cusp:- A double point on the curve is said to be cusp of both branches of the curve has same tangent at that point.



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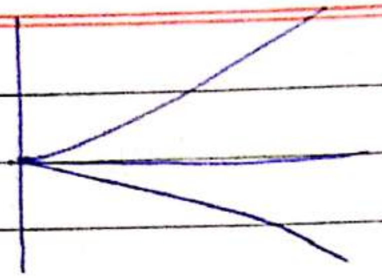
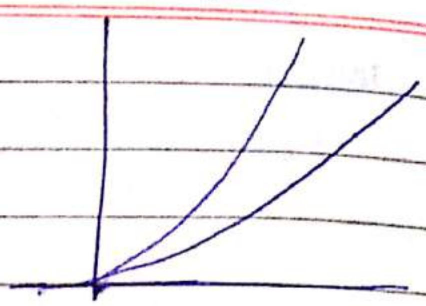


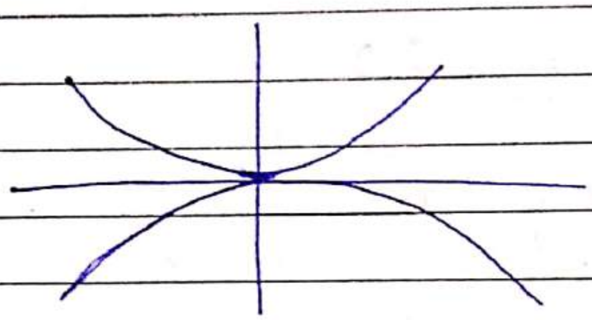
Single cusp of first species

b)



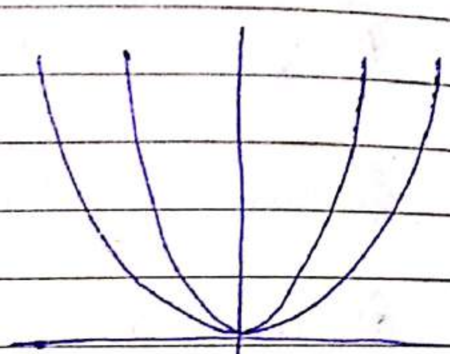
Single Cusp of Second species

(c)



Double Cusp of First Species

d)



Double cusp of second species

CONJUGATE POINT (Isolate Point) :-
A point on the curve is said to be conjugate if at that pt. tangent is Imaginary & there is no real points in their neighbourhood

<u>Necessary Conditions for the Existence of Double Pts.</u>
The necessary condition of a pt. (x,y) to be double pt. on the curve $f(x,y)=0$ is

$$F_x = \frac{\partial F}{\partial x} = 0 \qquad F_y = \frac{\partial F}{\partial y} = 0$$

Thus for finding double point the following eq :-

$$f_x = 0 \qquad f_y = 0 \quad \& \quad f = 0$$

Slope on the double point can be fixed out by the given eq

$$f_{xx}\left(\frac{dy}{dx}\right)^2 + f_{xy}\left(\frac{dy}{dx}\right) + f_{yy} = 0$$

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where
$f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial^2 y^2}$ $f_{xy} = \frac{\partial^2 f}{\partial y^2}$

A double point (x, y) will be Node if

① Node:- $[f^2_{xy} - 4 f_{xx} f_{yy}] > 0$

② cusp:- $f^2_{xy} - 4 f_{xx} f_{yy} = 0$

③ Conjugate :- $f^2_{xy} - 4 f_{xx} f_{yy} < 0$

Q Prove that the curve $y^2 = bx \tan\left(\frac{x}{a}\right)$ has a conjugate or node point at the origin
a and b haves unlike & like signs.

Let $f(x, y) = y^2 - bx \tan\left(\frac{x}{a}\right) = 0$

Now $\frac{\partial^2 f}{\partial x^2} = -\frac{b}{a} \sec^2\left(\frac{x}{a}\right) - \frac{b}{a} \sec^2 \frac{x}{a} - \frac{bx}{a^2} 2\left(\sec \frac{x}{a}\right)\left(\tan\frac{x}{a}\right)$

$\frac{\partial f}{\partial y} = 2y$ $\frac{\partial^2 f}{\partial y^2} = 2$ $\frac{\partial^2 f}{\partial xy} = 0$

for double pt.
Put $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore$ $(0,0)$ is a double point.
Now at origin $(0,0)$
$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = -\frac{b}{a} - \frac{b}{a} = -\frac{2b}{a}$

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$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2 \qquad \left.\frac{\partial^2 f}{\partial xy}\right|_{(0,0)} = 0$$

Now $b^2 - 4ac = (f(xy))^2 - 4 f_{xx} f_{yy} = 0 - 4\times 2\left(-\frac{2b}{a}\right)$

$= \frac{16b}{a}$ —— ①

Here RHS of ① > 0 if a and b have like signs &

⇒ (i) have 2 distinct real roots.

⇒ origin is node point.

Again RHS of ① < 0

if a and b have unlike signs

(i) have 2 complex roots.

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Curve Tracing

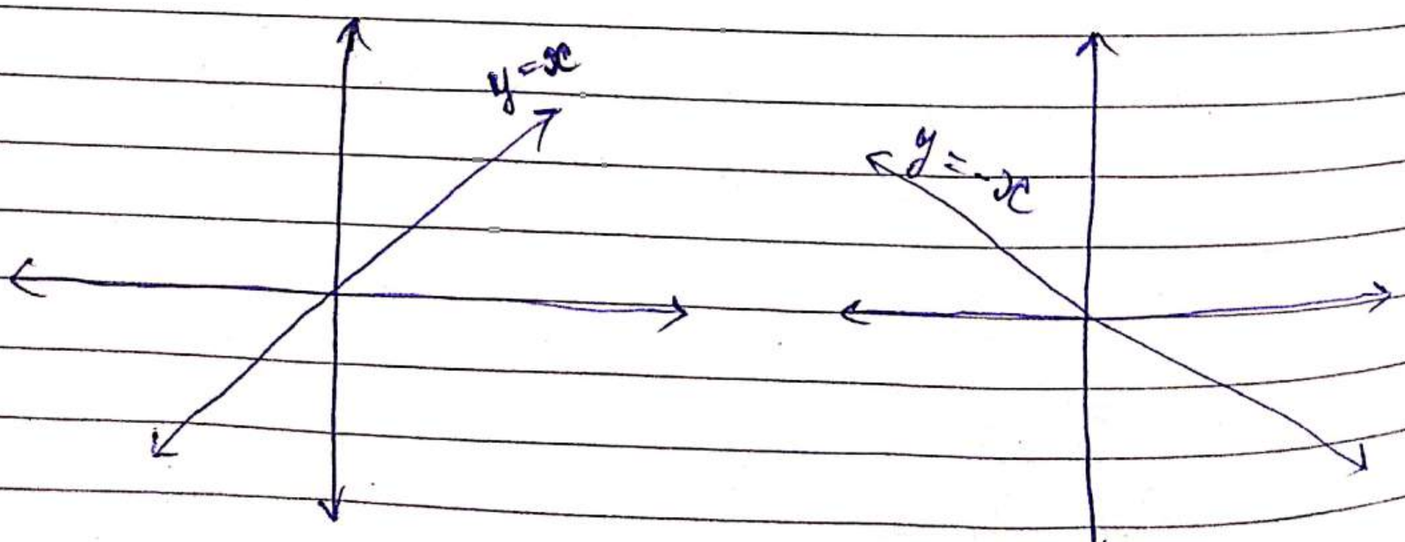
THEORY OF CARTESIAN CURVE:-

(1) **Symmetry:-** Let eq. of curve is $f(x,y)=0$

i) **Symmetry about X axis:-** If $f(x,y) = f(x,-y)$ i.e curve obtained equation has even power of y then the curve is symmetric about X axis.

ii) **Symmetry about Y axis:-** If $f(-x,y) = f(x,y)$ i.e. curve has even power of x then the curve is symmetric about y axis.

iii) **Symmetry about line $y=x$:-** If $f(y,x) = f(x,y)$ then the curve is symmetric about $y=x$.



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iv) Symmetric about line $y = -x$

If $f(-y, -x) = f(x, y)$ then the curve is symmetric about $y = -x$

(v) Symmetric about X and Y axis :-

If $f(-x, -y) = f(x, y)$ then the curve is symmetrical about both axis.

vi) Symmetry about Origin : If $f(-x, -y) = f(x, y)$ therefore the curve is symmetric in opposite quadrants.

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(2) <u>Origin</u>:- If the curve $f(x,y)=0$ has no constant term then curve passes through origin. Now for tangents, put lowest degree terms equate zero.
If curve has 2 distinct tangents then curve is <u>node</u> at origin.
And if curve has common tangent then curve is <u>cusp</u>.

(3) <u>Asymptotes</u> :- Find asymptote of the given curve using by earlier discussed methods.

(4) <u>Intersection Points</u> :- Find intersection points on X axis and Y axis by putting $y=0$ and $x=0$ respectively.

(5) <u>Region</u>:- Region of the curve can be found out by solving curve eq. in y which is form of x or in x which is form of y.
Then find the region for which x and y real.

Real function $f: R \to R$
Real valued function:- <u>Domain can be any but codomain should be real no.</u>
Range real no. का subset होती है।

(6) <u>Plotting of Points</u>
Increasing, if $\frac{dy}{dx} > 0 \quad \forall\ x \in [a,b]$

Decreasing, if $\frac{dy}{dx} < 0 \quad \forall\ x \in [a,b]$

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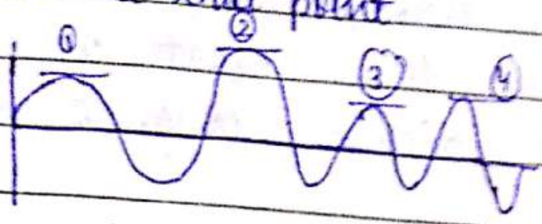
Maxima is a point
Maximum ⇒ Value at that point



1, 2, 3, 4 are local maxima.
2 is extreme maxima

Imp.

if $\frac{dy}{dx} = 0$ then check for maximum & minimum.

if $\left.\frac{d^2y}{dx^2}\right|_{x_0} > 0$ then x_0 is local minima

if $\left.\frac{d^2y}{dx^2}\right|_{x_0} < 0$ then x_0 is local maxima

Next if $\frac{d^2y}{dx^2} > 0$ $\forall x \in [a,b]$ then curve is convex

(∪ hold water)

if $\frac{d^2y}{dx^2} < 0$ ~~then go~~ $\forall x \in [a,b]$ then curve is CONCAVE

[∩ = spills water]

if $\left.\frac{d^2y}{dx^2}\right|_{x_0} = 0$ then go for Maxima or Minima or Point of inflexion.

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Example:- Trace the curve $y^2(a+x) = x^2(a-x)$

let $f(x,y) = y^2(a+x) - x^2(a-x) = 0$ —— ①

① <u>Symmetry</u>:- since $f(x,-y) = f(x,y)$
or since curve has even power of y therefore curve is symmetric about X axis.

② <u>Origin</u>:- Since curve has no constant term therefore curve passes through origin.

Since $f(x,y) = ay^2 + xy^2 - ax^2 + x^3 = 0$

$= x^3 + xy^2 + a(y^2 - x^2) = 0$ —— ②

For tangents at origin,
put lowest degree term in ② equal to zero

i.e. $a(y^2 - x^2) = 0$

$\Rightarrow y = \pm x$

$\therefore$ $y = \pm x$ are two distinct tangents that imply $(0,0)$ is a node.

③ <u>Asymptote</u>:- for parallel asymptote, put coefficient of highest power of x and y equal to zero.

$\therefore$ $x + a = 0$ is parallel asymptote to Y axis.

④ <u>Intersection Pts</u>:- For finding intersection pts. on X axis,
put $y=0$ in ①, we have

$x^2(a-x) = 0$

$x = 0, a$

put $x=0$ in ① we have $y^2 = 0$

Put $x = a$

So, $(0,0)$ and $(a,0)$ are intersection pts.

Intersection pts. on Y axis put $x=0$ in (1)

$$y = 0$$

we have $(0,0)$

$\therefore$ $(0,0)$ and $(a,0)$ are intersection points of curve on axis.

(5) **Region:** By (1)

$$y = \pm x\sqrt{\frac{a-x}{a+x}}$$

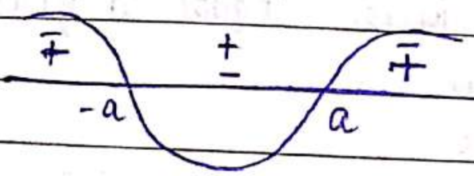
$a-x \longrightarrow x$ can't be $> a$

$a+x \longrightarrow x \neq -a$

x can't be $< a$

or by sign convention

$$y = \pm x\sqrt{\frac{-(x-a)}{x-(-a)}}$$



Since y is imaginary for $x > a$ & $x < -a$

$\therefore$ curve lies b/w $-a < x \leq a$

(6) **Plotting of Points:**

$$y^2(a+x) - x^2(a-x) = 0$$

$$y^2 = \frac{x^2(a-x)}{a+x}$$

$$2y\frac{dy}{dx} = \frac{[2x(a-x) - x^2]\{(a+x) - x^2(a-x)\}}{(a+x)^2}$$

$$= \frac{(2ax - 3x^2)(a+x)}{(a+x)^2} - x^2a + x^3$$

$$2y\frac{dy}{dx} = \frac{x(2a^2 - 3ax + 2ax - 3x^2 - xa + x^2)}{(a+x)^2}$$

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$$2y\frac{dy}{dx} = x\left[\frac{2a^2-2ax-2x^2}{(a+x)^2}\right]$$

$$y\frac{dy}{dx} = \frac{x\left[a^2-ax-x^2\right]}{(a+x)^2}$$

$$\frac{dy}{dx} = \frac{x}{y}\left[\frac{a^2-x^2-ax}{(a+x)^2}\right]$$

$$= \frac{x}{y}\left[\frac{a^2-x^2-ax}{(a+x)^2}\right]$$

for $y>0$ and $-a<x<0$

$(a+x)^2 > 0$ $\quad y>0$

$-a<x<0$

$\Rightarrow |x|<a$

$\Rightarrow |x|^2<a^2 \Rightarrow x^2<a^2$

$\Rightarrow a^2-x^2>0$

and

$-a<x<0$

$a>0 \Rightarrow -ax>0$

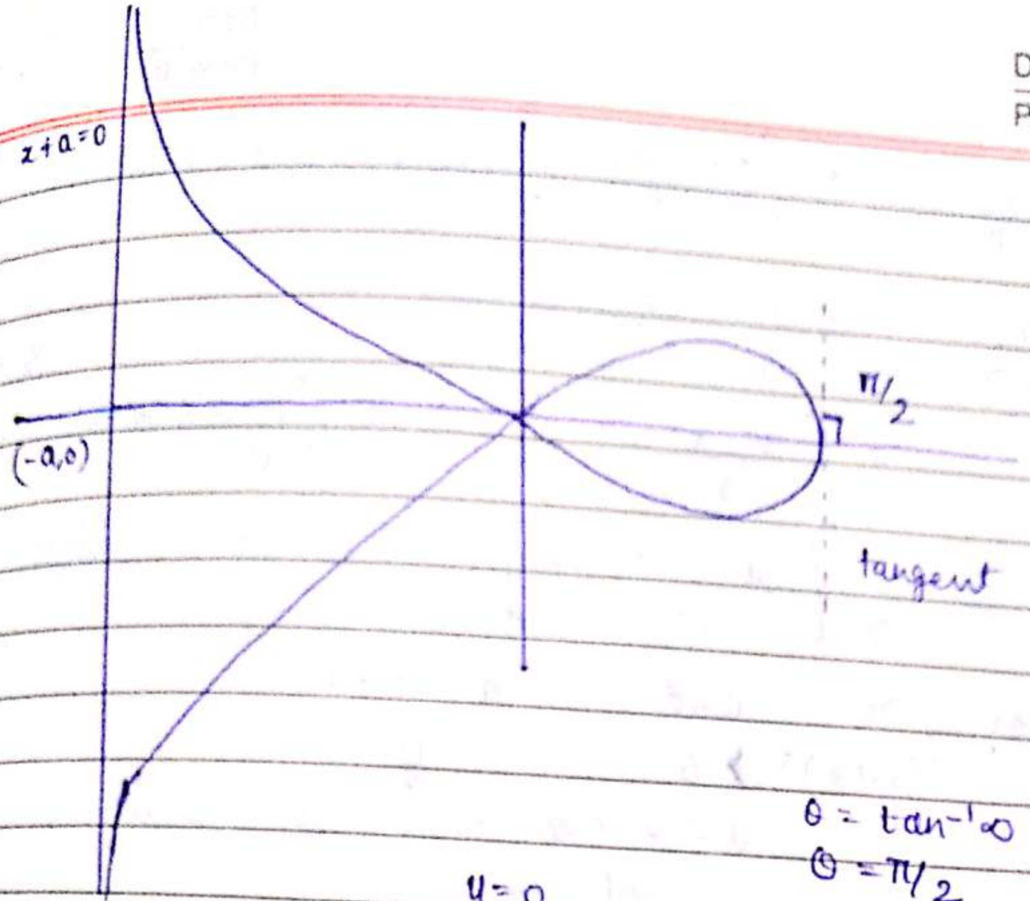
$\frac{dy}{dx}<0$ for $y>0$ $\quad -a<x<b$

~~Given curve is developing the patn~~

Given curve is decreasing in $-a<x<0$, $y>0$

Thus approx. shape is given as:-

P.T.O



$\theta = \tan^{-1}\infty$

$\theta = \pi/2$

$y = 0$

$x = a$

$\frac{dy}{dx} = \infty$

Q Trace the curve $y^2(2a-x) = x^3$

Let $f(x,y) = y^2(2a-x) - x^3 = 0$ —— ①

① Symmetry:- Since the curve eq. has even power of y, therefore curve is symmetric about x axis.

② Origin :- Since curve eq. has no constant terms, therefore curve passes through origin

Now the curve eq. can be written as

$$2ay^2 - xy^2 - x^3 = 0$$

For tangents at origin put lowest degree term equate zero

i.e. $y^2 = 0$

$\Rightarrow y, 0, 0$

Since curve has common tangent at origin (0,0)

∴ Origin is cusp.

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③ <u>ASYMPTOTES</u> : For parallel asymptotes, put coefficient of highest power of x and y equal to zero.

$x = 2a$ is parallel asymptote to Y axis.

$\because$ coefficient of x^3 is constant ~~as~~ So, parallel asymptotes to X axis does not exist.

④ <u>Region</u> :- Eq. ① can be written as

$$y^2 = \frac{x^3}{2a-x}$$

$$y = \pm x\sqrt{\frac{x}{2a-x}} = \pm x\sqrt{\frac{-x}{(x-2a)}}$$

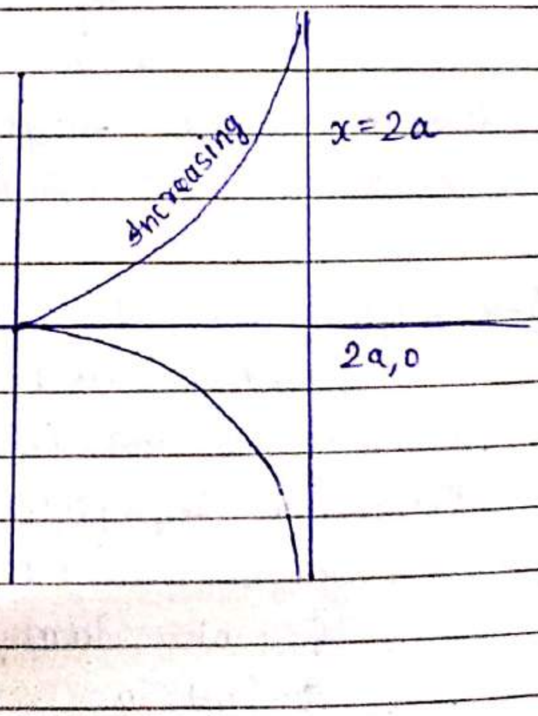
	−	+	−
	+	−	+
		0	2a

$\therefore$ y is imaginary for $x < 0$ & $x > 2a$

2a पर ∞ हो रहा है

Curve lies b/w

$0 \le x < 2a$



⑤ <u>Plotting of points</u>:

(Additional check total increasing or if there is any break between increasing)

$$\frac{dy}{dx} = \frac{(x^2)}{y}\,\frac{(3a-x)}{(2a-x)^2}$$ +

+

for $y > 0$ and $0 \le x$ $2a$

$$\frac{dy}{dx} > 0$$

$\Rightarrow$ curve is increasing in $0 \le x < 2a$ and $y \ge 0$

Thus approximate shape is as above :-

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Q Trace the curve $x^3+y^3=3axy$ Fallium of Descartes

$f(x,y)=x^3+y^3-3axy$ —— (1)

(1) **symmetry** : Since $f(x,y)=f(y,x)$ i.e.

since the eq. of the curve remains altered when y is replaced by x. Therefore the curve is symmetrical about the line $y=x$.

(2) **Origin** :- Since curve eq. has no constant terms, therefore curve passes through origin.

Now the given curve eq can be written as –

$$x^3+y^3-3axy=0$$

For tangents put lowest degree term in $f(x,y)$ equate to zero.

$$3axy=0 \qquad \cancel{x=0}$$

$$\Rightarrow \quad x=0 \qquad y=0$$

Since 2 distinct tangents so it is a node.

Since at origin 2 distinct tangents therefore origin is a node.

(3) **Asymptote** : $x^3+y^3=3axy$

Since coefficient of highest power of x & y is constant so there do not exist any parallel asymptote.

For oblique asymptotes, put $x=1$, $y=m$.

$$\phi_3(m)=m^3+1$$

$$\phi_2(m)=-3am$$

$$\phi_1(m)=0$$

For m, $\phi_3(m)=0=m^3+1$

$$\Rightarrow (m+1)(m^2-m+1)=0$$

$$\Rightarrow m=\frac{+1\pm\sqrt{3}\,i}{2}, \ -1$$

Only Real Asymptotes

for $m=-1$ $\quad c=-\frac{\phi_2(m)}{\phi_3(m)}=-\frac{(-3am)}{3m^2}=+\frac{a}{m}$

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$$c = -a$$

So, $y = -x - a$ ⇒ $x + y + a = 0$

(4) Intersection Pts :- To find intersection pts. on X axis, put $y=0$ in eq. (1)
$x = 0$ when $x=0$, $y=0$

so $(0,0)$ is the intersection point.

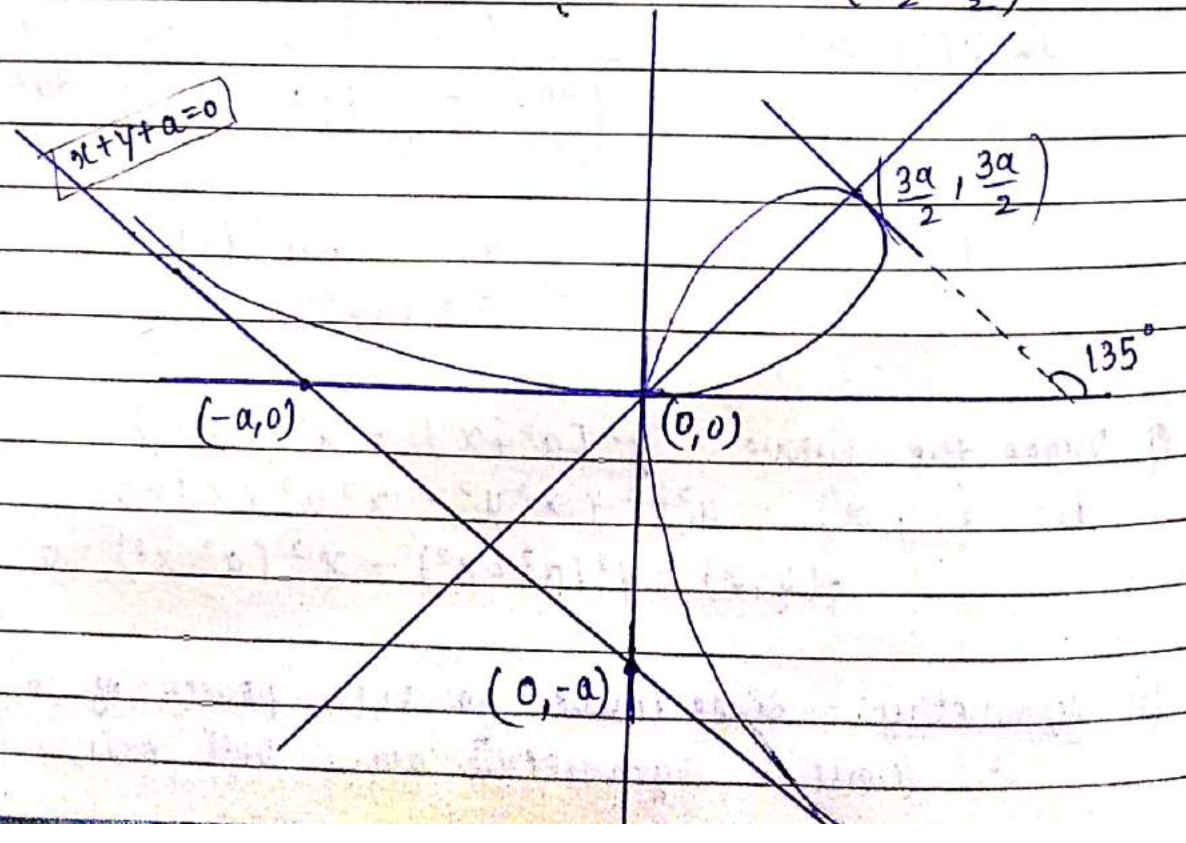
Intersection point on Y axis put $x=0$ in eq. (1)
$y=0$ we have $(0,0)$

⇒ $(0,0)$ is the intersection pt. of curve on the axis where curve pas

Again put $y=x$ in (1) we have.

$$2x^3 - 3ax^2 = 0$$
$$x^2(2x-3a) = 0$$
$$x = 0, 0 \qquad x = 3a/2$$
$$x = 0 \Rightarrow y = 0$$
$$x = \frac{3a}{2} \Rightarrow y = \frac{3a}{2}$$

∴ line $y=x$ intersect curve at $(0,0)$ and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$



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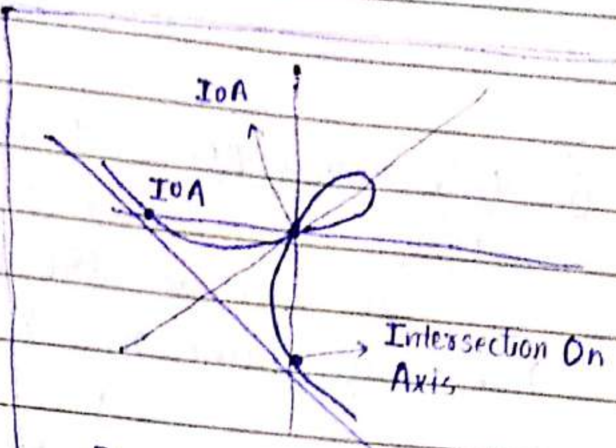
(5) **Region** :- Put $h = -k$ ~~or~~ $y = -k$ in (1)
we have

$-(h^3 + k^3) = 3ahk$

$\therefore a > 0 \Rightarrow$ LHS is $-ve$ &
RHS is $+ve$

which is not possible therefore curve doesn't pass in III Quadrant



For curve to be in III Quadrant there must be 3 IOA points but we have only origin as intersection point on axis. So curve does not lie in III Quadrant.

(6) Plotting of points
→ (झुकाव निकालने के लिए)

$x^3 + y^3 = 3axy$

$3x^2 + 3y^2 \dfrac{dy}{dx} = 3ay + 3ax\dfrac{dy}{dx}$

$\Rightarrow \dfrac{3(x^2 - ay)}{3(ax - y^2)} = \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - \left(\frac{3a}{2}\right)a} = \frac{-\frac{3a^2}{4}}{\frac{3a^2}{4}} = -1$$

$\tan\theta = -1 \qquad \Rightarrow \theta = \tan^{-1}(-1)$

$\Rightarrow \theta = 135°$

Q Trace the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

let $f(y, x) = a^2y^2 + x^2y^2 - x^2a^2 + x^4 = 0$

$f(y, x) = y^2(a^2 + x^2) - x^2(a^2 - x^2) = 0 \longleftarrow (1)$

(1) **Symmetry**:- since curve has even powers of x and y.
$\therefore$ curve is symmetric about both axis.

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② **Origin**:- since curve has ~~even powers of x and y~~ no constant terms therefore curve passes through origin.

For finding tangents at origin. put constant lowest degree term of ① equate zero.

$$\therefore \quad a^2(y^2-x^2)=0$$
$$y = \pm x$$

Since at origin, two distinct tangents so it is a node pt.

③ **Asymptote**:

$$y^2a^2+y^2x^2-x^2a^2+x^4=0$$
$$a^2+x^2=0$$
$$x=\pm ai \quad \rightarrow \text{complex root. So, no parallel asymptote parallel to } y \text{ axis.}$$

④ **Intersection points**: At $x=0$

$$y^2a^2=0 \quad \Rightarrow y=0$$
$$x^2(a^2-x^2) \quad \Rightarrow x=0 \text{ and } x=\pm a$$

Now $x=0 \Rightarrow y=0$

$x=a \Rightarrow y=0$ & $x=-a \Rightarrow y=0$

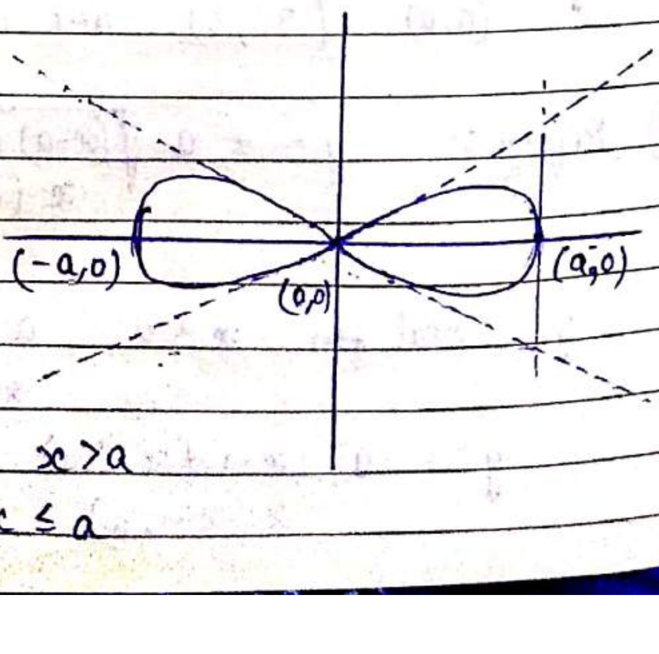
$\therefore$ $(0,0)$; $(a,0)$ $(-a,0)$ are intersection point of curve on Axes.

⑤ **Region**:- $y=\pm x\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ $\longrightarrow +ve$

$$y=\pm x\sqrt{\frac{-(x-a)(x+a)}{a^2+x^2}}$$

$\mp$ $\pm$ $\mp$

$-a$ a



y is imaginary for $x<-a$ and $x>a$

Curve lie b/w $-a \le x \le a$

Q Trace the curve $y^2 = \dfrac{a^2(x-a)(x-3a)}{x(x-2a)}$

let $f(x,y) = y^2 - \dfrac{a^2(x-a)(x-3a)}{x(x-2a)} = 0$ ——— (1)

(1) **Symmetry**:- since curve eq. has even power of y. therefore curve is symmetric about x axis.

(2) **Origin**:- $\because f(0,0) \neq 0$
$\therefore$ Curve does not pass through origin.

(3) **Asymptotes**:- $y^2(x)(x-2a) \Rightarrow a^2(x^2 - ax - 3a + 3a^2)$

$y^2x^2 - 2y^2xa = a^2x^2 - a^3x - 3a^3 + 3a^4$

So, $y^2 = +a^2$
$\Rightarrow y = \pm a$

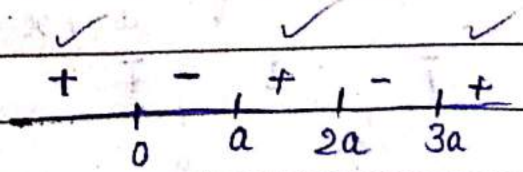
and $x^2 - 2ax = 0$
$x(x-2a) = 0$
$\Rightarrow x = 0, 2a$

(4) **Intersection Pts.** :- Eq (1) can be written as
$x(x-2a)y^2 = a^2(x-a)(x-3a) = 0$

Put $y=0$, we have
$x = a, 3a$

$\therefore$ $(a,0)$ $(3a,0)$ are intersection pts. on axis

(5) **Region**:- $y = \pm a\sqrt{\dfrac{(x-a)(x-3a)}{x(x-2a)}}$



y is real for $x < 0$, $a \leq x < 2a$
$x \geq 3a$

$y^2 = \dfrac{a^2(x-a)(x-3a)}{x(x-2a)}$

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Now shift origin at $(a,0)$

Put $x = X + a \qquad y = Y + 0$

$$\Rightarrow y^2 = \frac{a^2 [X] [X - 2a]}{[X+a] [X-a]}$$

$$\Rightarrow Y^2 [X+a][X-a] = a^2 X [X - 2a]$$

$$\Rightarrow Y^2X^2 - Y^2a^2 = a^2X^2 - 2a^3X$$

lowest degree term

$2a^3 X = 0$ tangent at new origin

$\Rightarrow X = 0$

$\Rightarrow x - a = 0$

$\Rightarrow x = +a$ will be tangent

Shift origin at $(3a,0)$

Put $x = X + 3a \qquad y = Y + 0$

$$Y^2 = \frac{a^2 [X+2a][X]}{[X+3a] [X+a]}$$

$$\Rightarrow Y^2X^2 + 4aXY^2 + 3a^2Y^2 = a^2X^2 + 2a^3X$$

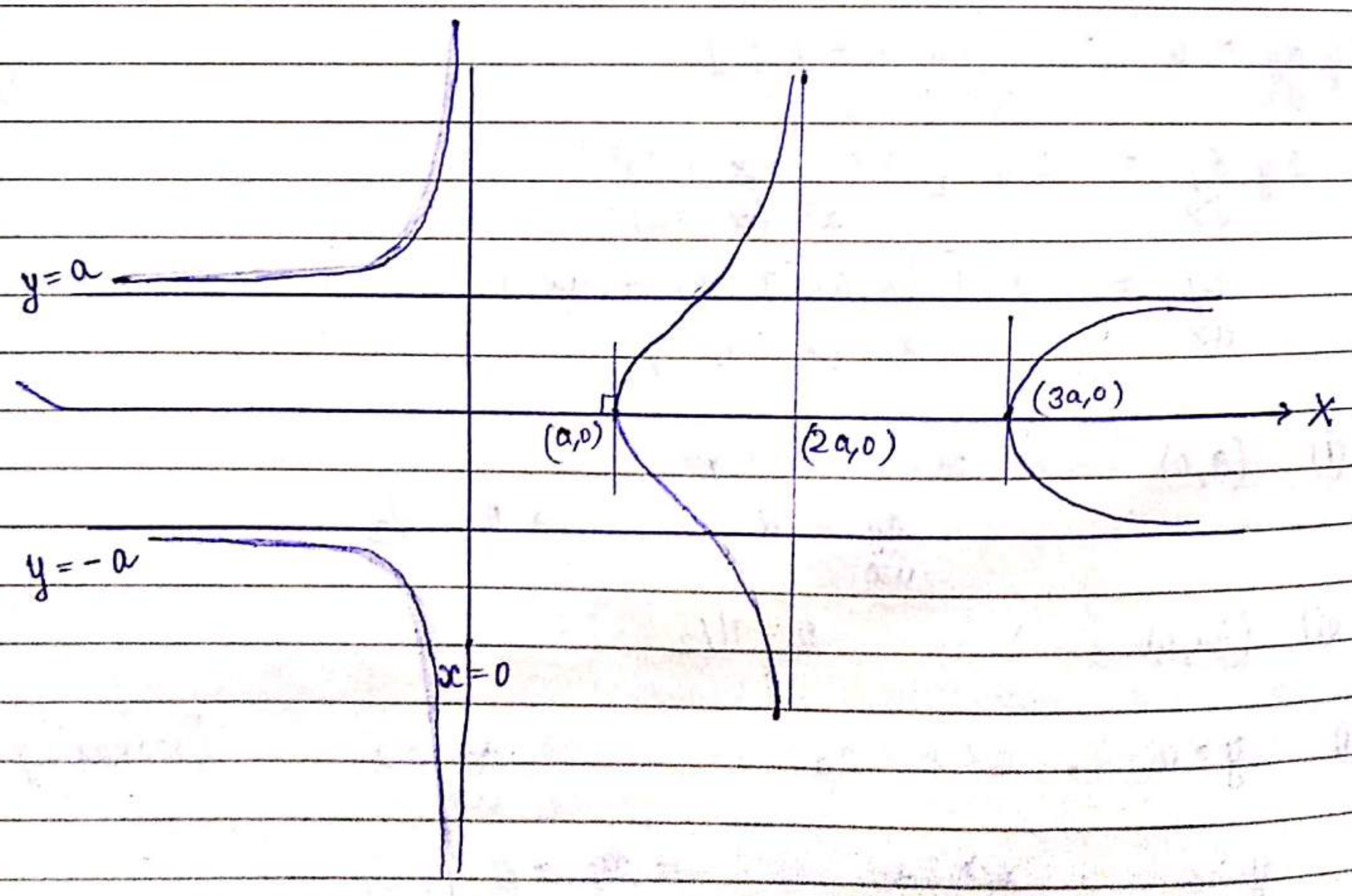
lowest degree term for Tangent

$2a^3 X = 0 \quad \Rightarrow X = 0$

So, $x - 3a = 0$

$x = 3a$

will be tangent.



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⑥ Plotting of points :

$$y^2 = a^2 \frac{(x-a)(x-3a)}{x(x-2a)}$$

$$2y\frac{dy}{dx} = a^2\left[\{-a(x-3a) + (-3a)(x-a)\}\{x(x-2a)\} - \{x-2a\right.$$

$$y^2 = \frac{a^2(x^2-4ax+3a^2)}{x^2-2ax}$$

$$2y\frac{dy}{dx} = a^2\frac{[(2x-4a)(x^2-2ax) - (2x-2a)(x^2-4ax+3a^2)]}{(x^2-2ax)^2}$$

$$= a^2\frac{[2x^3-4ax^2-4ax^2+8a^2x-2x^3+8ax^2-6a^2x+2a^2x^2-8a^2x+6a^3]}{(x^2-2ax)^2}$$

$$2y\frac{dy}{dx} = a^2\frac{[2ax-6a^2x+6a^3]}{x^2(x-2a)^2}$$

$$2y\frac{dy}{dx} = 2a^3\frac{[x^2-3ax+3a^2]}{x^2(x-2a)^2}$$

$$\frac{dy}{dx} = \frac{a^3[(x-a)(x-3a)+ax]}{x^2(x-2a)^2\,y}$$

(i) $(a,0)$ $\quad x=a \Rightarrow y=0$

$\frac{dy}{dx} = \infty \quad \Rightarrow \theta = \pi/2$

(ii) $(3a,0)$ $\quad \theta = \pi/2$

(i) $y>0$, $a<x<2a$ $\quad \Rightarrow \frac{dy}{dx} > 0$ Increasing

$y>0$, $x>3a$ $\quad \Rightarrow \frac{dy}{dx} > 0$ Increasing

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Q Trace the curve

$$ay^2 = x^2(a-x)$$

Let $f(x,y) = ay^2 - x^2(a-x)$ ——— (1)

(1) Symmetry:- Since curve eq: has even power of y
∴ Curve is symmetric about X axis.

(2) Origin:- ∵ $f(0,0) = 0$
∴ Curve passes through Origin.

$ay^2 - ax^2 = 0$ (lowest degree term = 0)

$y = \pm x$

Since at origin 2 distinct tangents
⇒ Origin is a node point

(3) Asymptotes:- $ay^2 - x^2a + x^3 = 0$
No parallel Asymptotes.

For Oblique asymptote,

$$\phi_3(m) = 1$$

$$\phi_2(m) = am^2 - a$$

$$\phi_1(m) = 0$$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{\phi_2(m)}{0} = \infty$$

So, no real asymptotes exist.

(4) Intersection Points:- Intersection on X axis put $y=0$

$$x^2(x-a) = 0$$

$$x = a, 0, 0$$

$(0,0)$ $(a,0)$ Intersection pts.

if $x=0$, $y=0$

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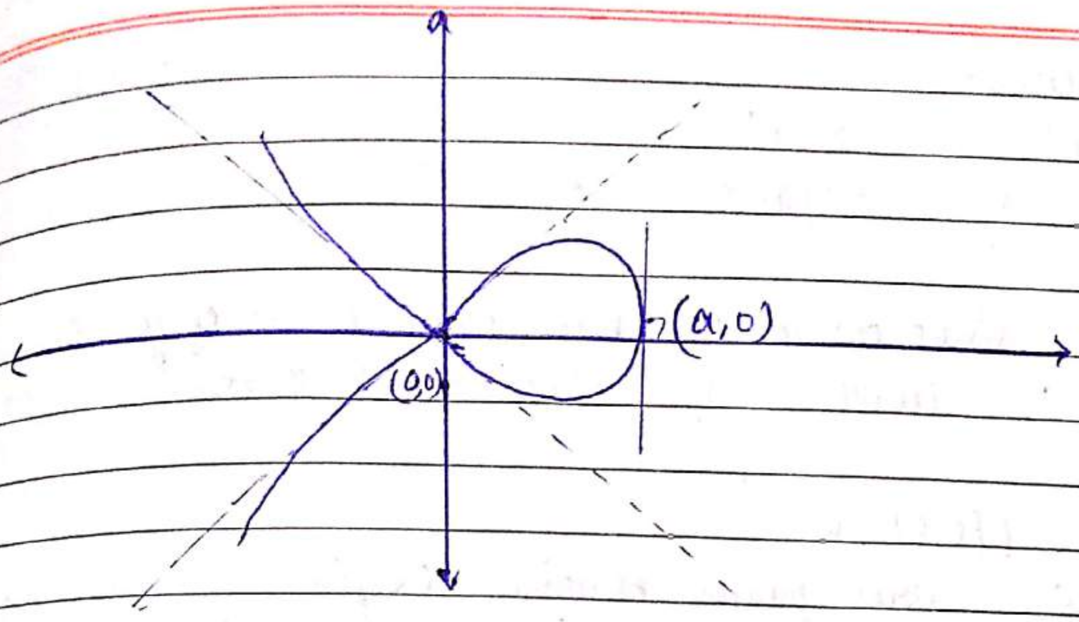


Eq. ① can be written as

$$y = x\sqrt{\frac{a-x}{a}} = x\sqrt{\frac{-(x-a)}{a}}$$

y will be real for $x < a$

Plotting of Points:- $ay^2 = x^2(a-x)$

$$2ay\frac{dy}{dx} = 2x(a-x) + x^2(-1)$$

$$= -2x^2 + 2ax - x^2$$

$$\frac{dy}{dx} = \frac{2ax - 3x^2}{2ay}$$

at $(a,0)$ $\theta = \tan^{-1}\infty$

$\theta = \pi/2$

Curvature Tracing for Polar Curves:

Procedure for tracing the curves with polar equation

Let curve eq. $f(r,\theta) = 0$

① SYMMETRY :-

a) If $f(r,-\theta) = f(r,\theta)$ then curve is symmetric about initial line $\theta=0$

b) If $f(r,\pi-\theta) = f(r,\theta)$ then curve is symmetric about $\theta = \pi/2$.

c) If $f(-r,\theta) = f(r,\theta)$

or $f(r,\pi+\theta) = f(r,\theta)$ then curve is symmetric about pole.

d) If $f(r, \frac{3\pi}{2}-\theta) = f(r,\theta)$ then curve is symmetric about $\theta = \frac{3\pi}{4}$

② POLE :-

If $r = f(\theta_1) = 0$ for $\theta = \theta_1$

then curve passes through pole and has tangents at pole is $\theta = \theta_1$

③ ASYMPTOTE :-

If $\lim_{\theta \to \theta_1} r = \infty$ then the curve $r = \frac{1}{f(\theta)}$ has asymptotes and is

given by $r \sin(\theta-\theta_1) = f'(\theta_1)$ where $\theta = \theta_1$ is solution of $f(\theta)=0$

④ POINTS ON THE CURVE :-

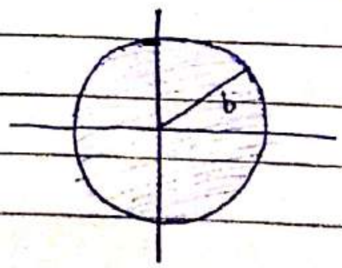
Find r for some value of θ by curve eq.

θ	0	$\pi/4$	$\pi/2$		
r					

⑤ REGION :-

a) If r is imaginary for values of θ then curve does not exist for that value of θ.

b) If $r_{max.} = b$ for values of θ then curve lie inside a circle of radius b.



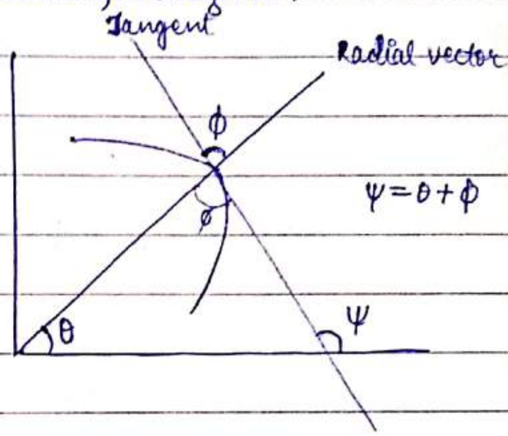
⑥ DIRECTION OF TANGENTS :-

Slope of curve at any point (r, θ) can be found out by following eq.

$$\tan \phi = r \frac{d\theta}{dr}$$

where ϕ is angle b/w radial vector & tangent.

and $\theta + \phi = \psi$



⑦ LOOP :-

If curve eq. are in the form of

$r = a \sin n\theta$

or $r = a \cos n\theta$

then curve has n or $2n$ loops if n is odd or even respectively.

Q Trace the curve $r^2 = a^2 \sin 2\theta$ — Lemniscate

let $f(r, \theta) = r^2 - a^2 \sin 2\theta = 0$ ——— ①

① SYMMETRY:- Since $f(-r, \theta) = f(r, \theta)$ then curve is symmetrical about pole.

② POLE :- $r^2 = a^2 \sin 2\theta = 0$

$\Rightarrow \sin 2\theta = 0$

$\Rightarrow 2\theta = n\pi$

$\Rightarrow \theta = \frac{n\pi}{2}$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

$\Rightarrow \theta = 0, \pi/2$

Eq. of tangent at pole are $\theta = 0$ and $\theta = \pi/2$

Both are distinct. $\Rightarrow$ Pole is node.

③ ASYMPTOTE :- Since r has finite values for all θ.

$\therefore$ Curve does not have asymptotes.

④ POINTS ON THE CURVE :- $r^2 = a^2 \sin 2\theta \Rightarrow r = \pm a\sqrt{\sin 2\theta}$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$
r	0	$\pm a\sqrt{\frac{\sqrt{3}}{2}}$	a	$\pm a\sqrt{\frac{\sqrt{3}}{2}}$	0	$\pm a\sqrt{\frac{-\sqrt{3}}{2}}$	$\pm a\sqrt{-1}$

⑤ REGION :- $\because$ max. value of $\sin 2\theta = 1$

$\therefore r_{max} = a$

$\therefore$ curve lies in 'a' radius circle

since for $\frac{\pi}{2} < \theta < \pi$, $\frac{3\pi}{2} < \theta < 2\pi$

has imaginary values

$\Rightarrow$ curve has no branch in II & IV quadrant

⑥ DIRECTION OF TANGENT :-

$$r^2 = a^2 \sin 2\theta$$

$$2r = a^2 \cos 2\theta \frac{d\theta}{dr}$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{2r \quad a^2 \sin 2\theta}{2r\, a^2 \cos 2\theta}$$

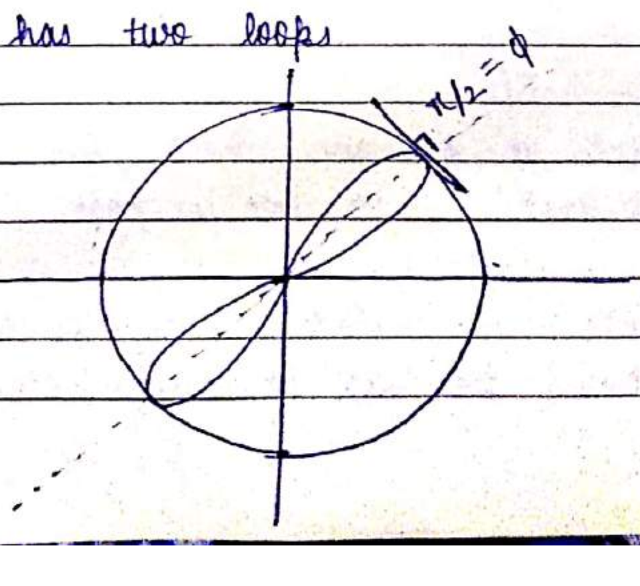
$$\tan\phi = \tan 2\theta$$

$$\phi = 2\theta$$

$\theta = 0 \Rightarrow \phi = 0$

$\theta = \frac{\pi}{2} \Rightarrow \phi = \pi$

⑦ LOOP :- curve has two loops

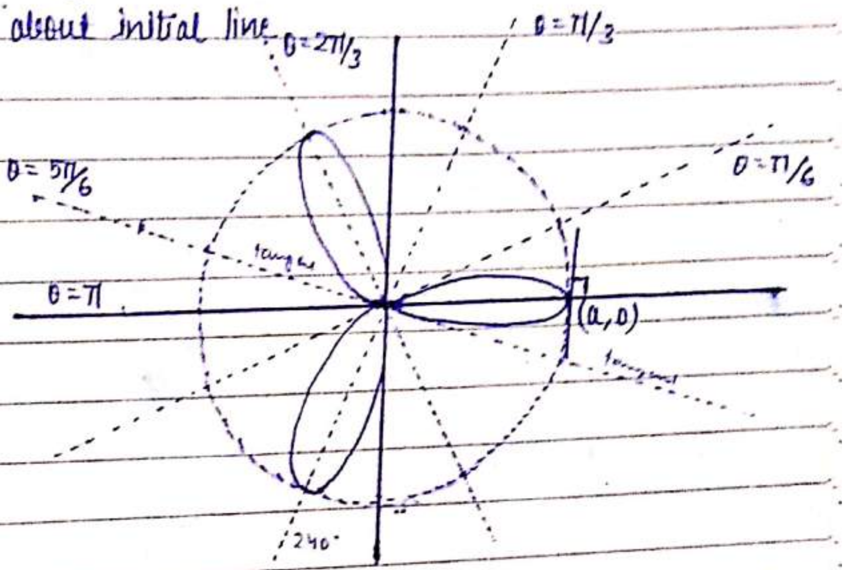


Q Trace the curve $r = a\cos 3\theta$ (Three leaved Rose)

Let $f(r,\theta) = r - a\cos 3\theta$ ——— ①

① SYMMETRY:- Since $f(r,-\theta) = r\cos(-3\theta)$
$= r\cos(3\theta)$
$= f(r,\theta)$

curve is symmetric about initial line



② POLE: $r = 0$

$a\cos 3\theta = 0$

$\cos 3\theta = 0$

$3\theta = (2n+1)\dfrac{\pi}{2}$

$\theta = (2n+1)\dfrac{\pi}{6}$

$\Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

Therefore, $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ are 3 distinct tangents on plane $\Rightarrow$ Three branches of curve

③ ASYMPTOTES: Since $r = a\cos\theta$ gives finite values for all θ
$\Rightarrow$ there is no asymptote b/w $-1 \le \cos 3\theta \le 1$

④ POINTS ON THE CURVE

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$5\pi/6$	$3\pi/2$	π
r	a	0	$-a/\sqrt{2}$	$-a$	0	a	0	$-a$

⑤ REGION: since $|\cos 3\theta| \le 1$ $\Rightarrow r_{max.} = a$
Thus, curve lies inside a circle of radius 'a'

⑥ DIRECTION OF TANGENTS:

$r = a\cos 3\theta$

$1 = -a(3\sin 3\theta)\dfrac{d\theta}{dr}$

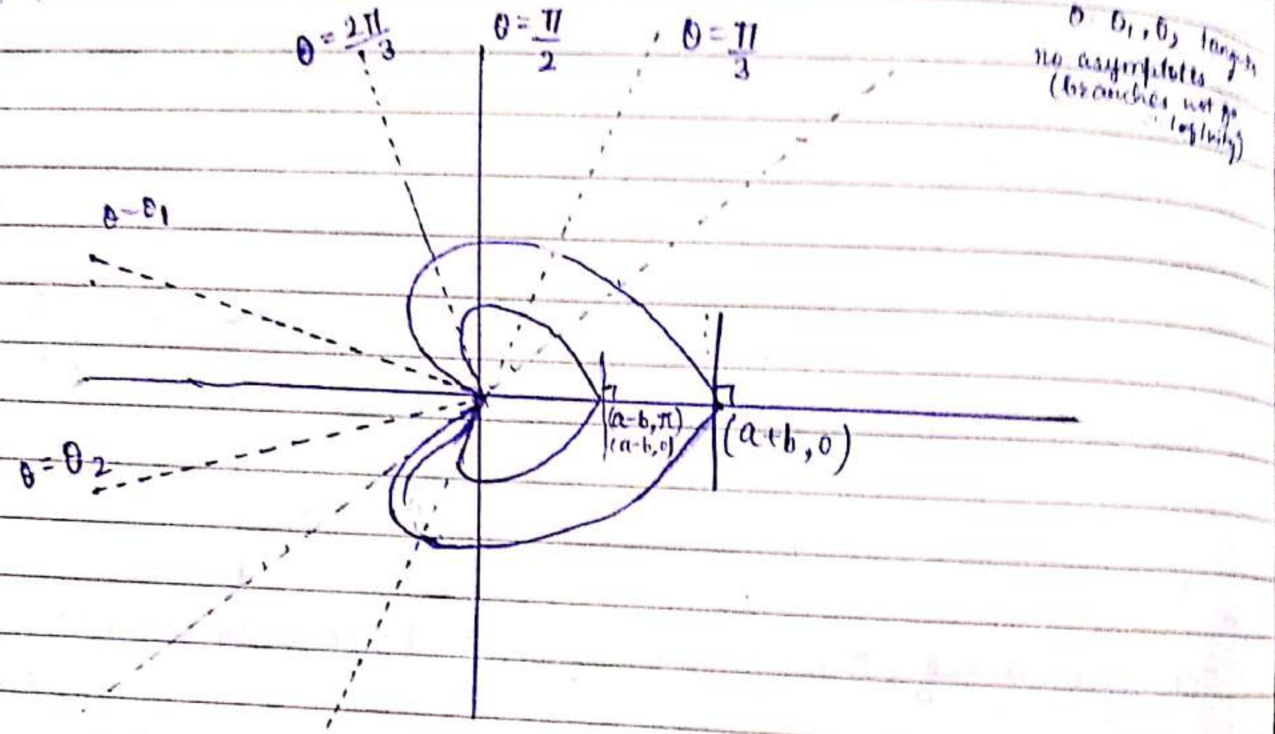
$\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{-a\cos 3\theta}{3a\sin 3\theta} = \dfrac{\cos 3\theta}{3\sin 3\theta}$

at $\theta = 0$ $\tan\phi = \dfrac{1}{0} = \infty \Rightarrow \phi = 90°$

(7) LOOP :- ∵ curve eq. is $r = a\cos 3\theta$ & 3 is odd no.
$\Rightarrow$ Curve has 3 loops.

Q. Trace the curve $r = a + b\cos\theta$ $a < b$ (Limacon)

$f(r,\theta) = r - a - b\cos\theta$ ——— (i)



(1) SYMMETRY :- $f(r,-\theta) = r - a - b\cos(-\theta)$

$= r - a - b\cos\theta = f(r,\theta)$

∴ Curve is symmetric about initial line.

(2) POLE :- $r = 0$

$\Rightarrow a + b\cos\theta = 0$

$\Rightarrow \cos\theta = -\frac{a}{b}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{a}{b}\right)$

Since $\cos\theta$ has -ve values in II and III quadrant.
Therefore $\cos^{-1}\left(\frac{-a}{b}\right)$ has 2 distinct values.
say $\theta = \theta_1$ & $\theta = \theta_2$

$\Rightarrow$ Pole is Node.

③ ASYMPTOTE : since $r = a + b\cos\theta$ has finite values for all θ
$\Rightarrow$ curve has no asymptote.

④ POINTS ON THE CURVE :- $r = a + b\cos\theta$

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	θ_1	$\theta_1 < \pi$	π
r	$a+b$	$a+\frac{b}{\sqrt{2}}$	$a+\frac{b}{2}$	a	$a-b/2$	0	$-ve$	$a-b$

⑤ REGION :- $r_{max} = a+b$
$r_{min} = a-b$ $\Rightarrow$ curves lies inside a circle of radius $a+b$.

⑥ DIRECTION OF TANGENTS

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\frac{dr}{d\theta} = -b\sin\theta$$

$$\tan\phi = \frac{a+b\cos\theta}{a-b\sin\theta}$$

$$\tan\phi = \infty \Rightarrow \phi = \pi/2$$

At $\theta = \pi$

$$\Rightarrow \tan\phi = \infty$$
$$\Rightarrow \phi = \pi/2$$

⑦ LOOP :- FOR SOME values of θ, r becomes $-ve$. ~~since~~
since $a < b$
$\Rightarrow a - b < 0$
$\therefore$ there is an inner loop between $\theta = \theta_1$ & $\theta = \theta_2$

Q. Trace the curve $r = ae^{m\theta}$ — ① equiangular spiral

Let $f(r,\theta) = r - ae^{m\theta}$

① $\boxed{\text{SYMMETRY}}$ since curve $f(r,-\theta) \neq f(r,\theta)$

$f(r, \pi-\theta) \neq f(r,\theta)$

$f(r, \frac{3\pi}{2}-\theta) \neq f(r,\theta)$

$f(r, \pi+\theta) \neq f(r,\theta)$

$\therefore$ curve is not symmetric.

② $\boxed{\text{POLE}}$ $\because e^{\theta} > 0 \quad \forall \theta$

$\Rightarrow r > 0 \quad \forall \theta$

$\therefore$ Curve does not ~~pass~~ passes through pole.

③ $\boxed{\text{POINTS ON THE CURVE}}$

θ	0	$\pi/2$	π	2π	3π
r	a	$ae^{m\pi/2}$	$ae^{m\pi}$	$ae^{m2\pi}$	$ae^{3m\pi}$

④ $\boxed{\text{REGION}}$ r exist for all value of θ.

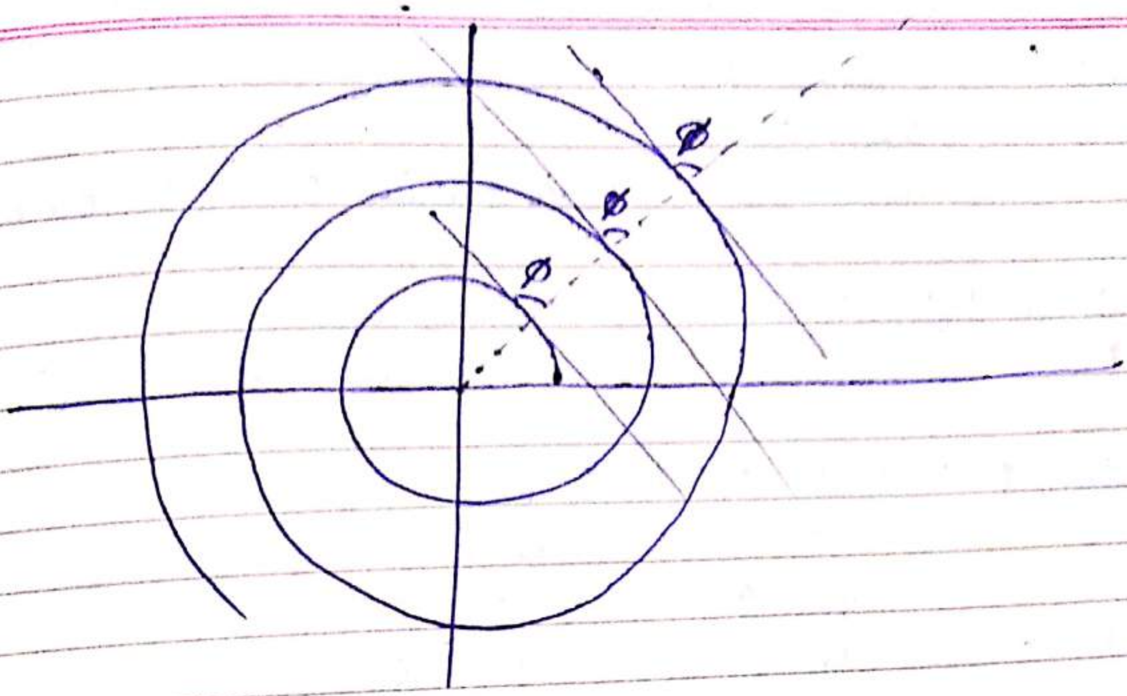
⑤ $\boxed{\text{DIRECTION OF TANGENTS}}$

$\frac{d\theta}{dr} \Rightarrow$ $r = ae^{m\theta}$

$1 = ame^{m\theta}\frac{d\theta}{dr}$

$\tan\phi = \frac{ae^{m\theta}}{ame^{\theta}} = \frac{1}{m}$ which depend on m and therefore direction of tangents at each point of θ is constant.

⑥ $\boxed{\text{LOOP}}$ There is no loop in curve because for each θ, r is different.



Q. Trace the curve $\quad r = a+b \cos\theta \quad (a>b)$

<u>SYMMETRY</u> :- Let $f(r,\theta) = r - a - b\cos\theta$

$\because f(r,-\theta) = f(r,\theta)$

$\therefore$ Curve is symmetrical about initial line.

<u>POLE</u> :- $r=0 \Rightarrow a+b\cos\theta = 0$

$\Rightarrow \cos\theta = -\dfrac{a}{b}$

$\therefore |\cos\theta| > 1$ [as $a>b$ (given)]

which is not possible. Hence for no value of θ, r is equal to zero. Therefore curve does not pass through pole.

<u>ASYMPTOTE</u>: Since $r = a+b\cos\theta$ has finite values for all θ

$\Rightarrow$ Curve has no asymptotes.

<u>POINTS ON THE CURVE</u> :-

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$a+b$	$a+b/\sqrt{2}$	$a+b/2$	a	$a-b/2$	$a-b$ +ve

Curve

REGION: $r_{max} = a+b$

$r_{min} = a-b$

$\therefore$ Curve lies in a radius of circle $a+b$

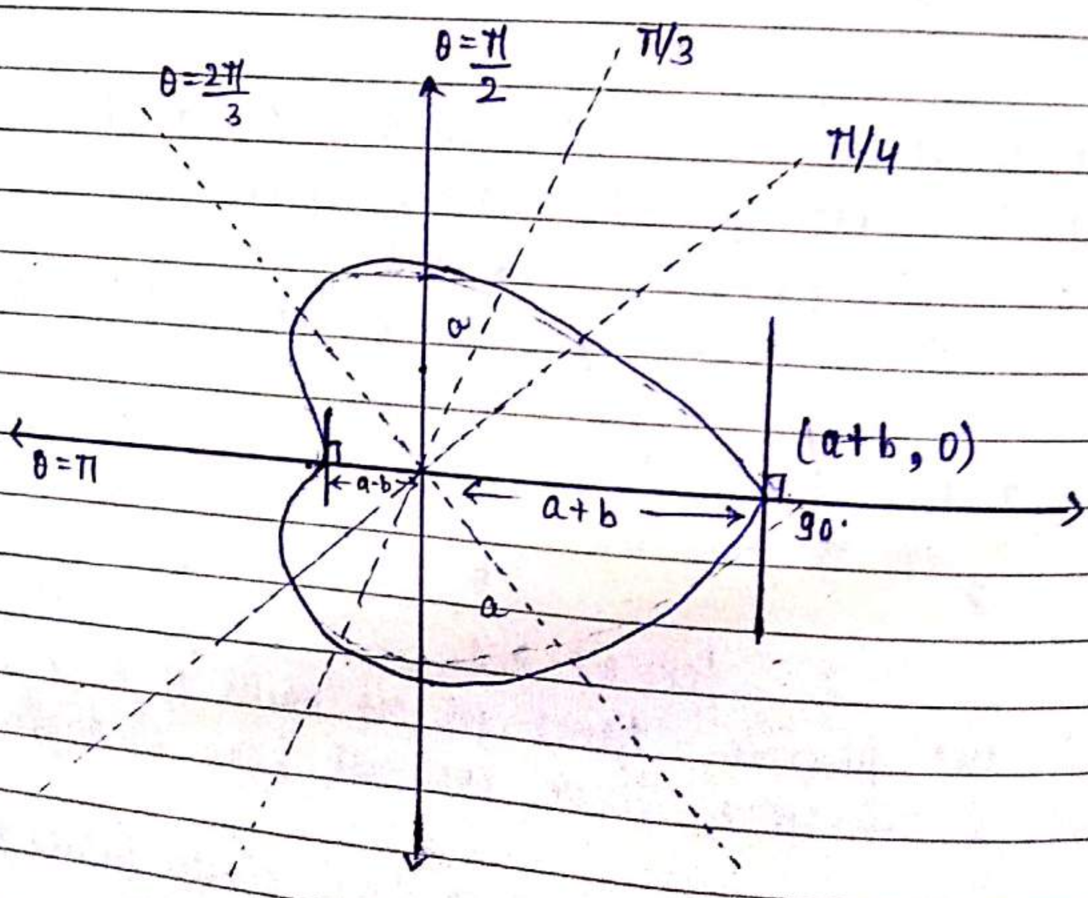
DIRECTION OF TANGENTS: $r = a + b\cos\theta$

$$1 = -b\sin\theta \frac{d\theta}{dr}$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a+b\cos\theta}{-b\sin\theta}$$

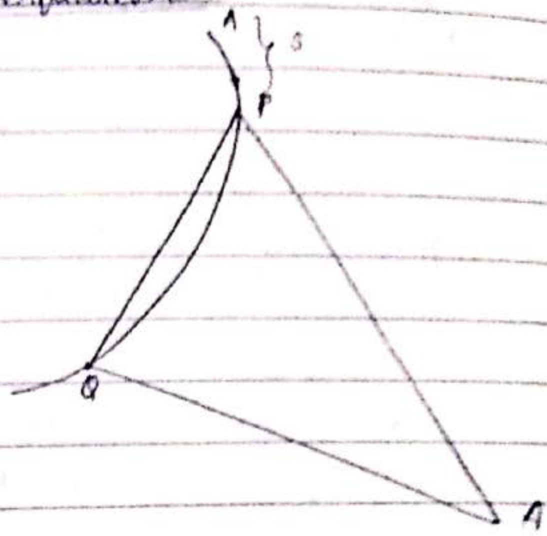
at $\theta = 0^\circ$ $\tan\phi = \infty \Rightarrow \phi = 90^\circ$

$\theta = \pi$ $\tan\phi = \infty \Rightarrow \phi = \pi/2$



Pedal Equation

Derivative of an arc and Pedal Equation



DERIVATIVE OF AN ARC: If we consider length of an arc from a fixed point A is s and which is function of x

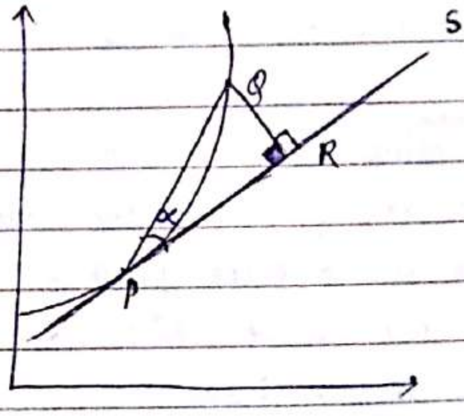
i.e. $s = f(x)$.

Also let two points P and Q on the curve such that arc PQ is concave towards chord PQ.

Chord PQ < arc PQ

< PA + QA

where PA and QA are two lines which cover the arc PQ.



Ratio of an arc with its chord

Let two points on the curve such that arc PQ is always concave towards chord PQ.

Also, that a straight line PS is the tangent on the curve at point P.

Now, draw a perpendicular on PS from Q

Let $\angle QPR = \alpha$

$\therefore$ chord PQ < arc PQ

< PR + QR ——— ①

Now by ΔPQR

PR = chord PQ. $\cos\alpha$

QR = chord PQ. $\sin\alpha$

Putting values in ①

chord PQ < arc PQ < chord PQ. $\cos\alpha$ + chord PQ. $\sin\alpha$

$$\Rightarrow 1 < \frac{\text{arc } PQ}{\text{chord } PQ} < \cos\alpha + \sin\alpha$$

As $Q \to P \Rightarrow \alpha \to 0$ and

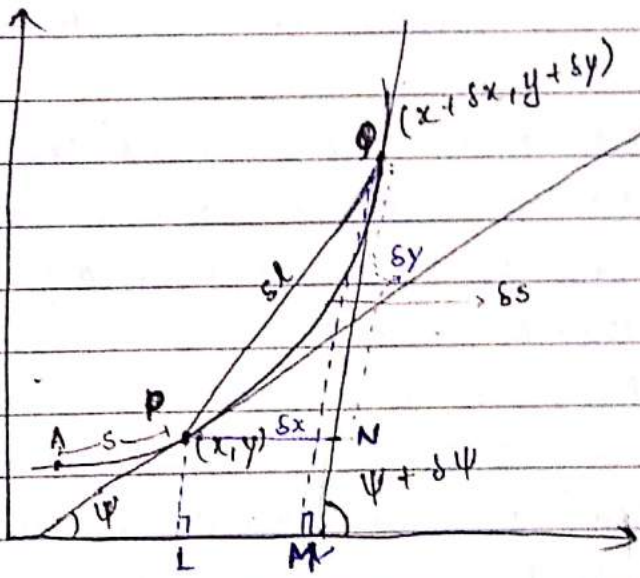
PQ $\to$ tangent at P.

$$\lim_{Q\to P} 1 < \lim_{Q\to P} \frac{\text{arc } PQ}{\text{chord } PQ}$$

$$< \lim_{\alpha\to 0} \underbrace{\cos\alpha + \sin\alpha}_{=1}$$

$$1 < \lim_{Q\to P} \frac{\text{arc } PQ}{\text{chord } PQ} < 1$$

$$\Rightarrow \boxed{\lim_{Q\to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1}$$



Derivative of an Arc:

1) Cartesian formula:-

Let the curve eq. is

$y = f(x)$

nearest

~~Two~~ points $P(x, y)$ and

$Q(x+\delta x, y+\delta y)$ on the curve.

Also let a fixed point A on the curve which has arc length from P is s.

and arc length from Q is $s + \delta s$

Therefore, length of arc $PQ = \delta s$

Let length of chord $PQ = \delta l$

Now, draw two tangents on P and Q, ~~reptac~~ which makes angle ψ and $\psi + \delta\psi$ with x-axis respectively.

Again, draw two perpendicular PL and QM on X-axis from P and Q respectively

Therefore, $PN = OM - OL$

$= x + \delta x - x = \delta x$

and $QN = QM - MN = QM - PL$

$= y + \delta y - y = \delta y$

Now, in ΔPQN

$$PQ^2 = PN^2 + QN^2$$

$$\Rightarrow \delta l^2 = \delta x^2 + \delta y^2 \qquad \text{———} \star$$

$$\Rightarrow \left(\frac{\delta l}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \qquad \text{———} (1)$$

Now $\frac{\delta s}{\delta x} = \frac{\delta s}{\delta l} \cdot \frac{\delta l}{\delta x}$

From (1), $\frac{\delta s}{\delta x} = \pm \frac{\delta s}{\delta l}\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \qquad \text{———} (2)$

As $Q \to P$ $\quad \delta x \to 0 \qquad \text{———} (3)$

$\therefore \lim_{Q \to P} \frac{\delta s}{\delta l} = 1.$

We know $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

Therefore (2) becomes

$$\lim_{\delta x \to 0} \frac{\delta s}{\delta x} = \pm \sqrt{1 + \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)^2}$$

$$\frac{ds}{dx} = \pm \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

If s increases when x increases then positive +ve sign otherwise take -ve sign.

$$\frac{ds}{dy} = \pm \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Corr. If curve eq.n is $x = f(y)$

Now by eq ($\star$), divide it by $(\delta y)^2$, we have

$$\left(\frac{\delta l}{\delta y}\right)^2 = \left(\frac{\delta x}{\delta y}\right) + 1$$

$$\frac{\delta s}{\delta y} = \frac{\delta s}{\delta l} \cdot \frac{\delta l}{\delta y}$$

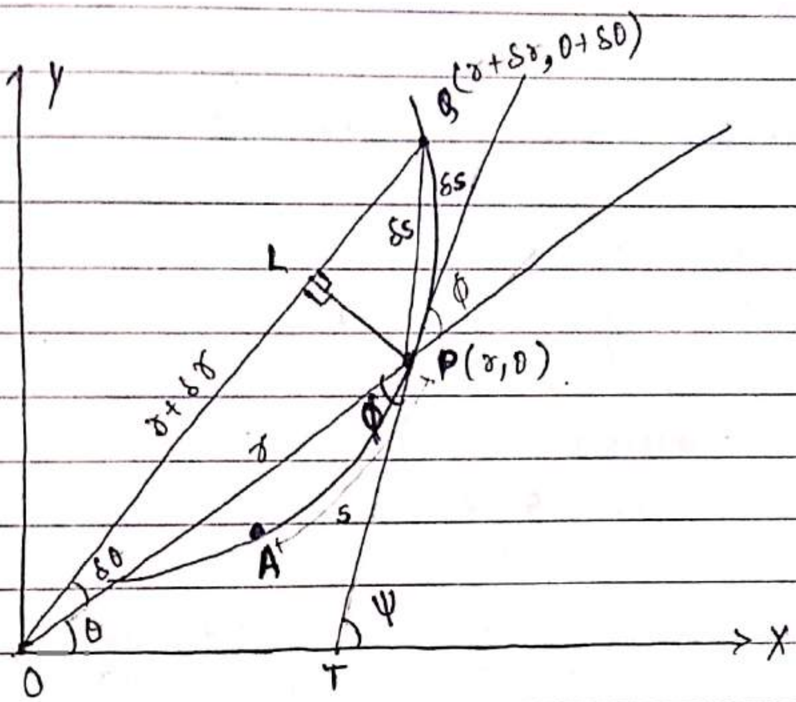
$$\frac{\delta s}{\delta y} = \pm \frac{\delta s}{\delta l}\sqrt{1 + \left(\frac{\delta x}{\delta y}\right)^2}$$

As $Q \to P$ then $\delta y \to 0$

$$\lim_{\delta y \to 0} \frac{\delta s}{\delta y} = \pm \sqrt{1 + \lim_{\delta y \to 0} \left(\frac{\delta x}{\delta y}\right)^2} \qquad \because \lim_{Q \to P} \frac{\delta s}{\delta Q} = 1$$

$$\frac{ds}{dy} = \pm \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Polar Form



Let curve eq. $r = f(\theta)$ —— (1)

Now $\frac{ds}{d\theta} = \lim_{\delta\theta \to 0} \frac{\delta s}{\delta\theta}$

$$= \lim_{\delta\theta \to 0} \frac{\text{arc } PQ}{\delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{\text{arc } PQ}{\text{chord } PQ} \cdot \frac{\text{chord } PQ}{\delta\theta}$$

$$\frac{ds}{d\theta} = \lim_{\delta\theta \to 0} \frac{\text{arc } PQ}{\text{chord } PQ} \lim_{\delta\theta \to 0} \frac{\text{chord } PQ}{\delta\theta} \quad \text{—— (2)}$$

as $\delta\theta \to 0 \Rightarrow Q \to P$

$$\lim_{\delta\theta \to 0} \frac{\text{arc } PQ}{\text{chord } PQ} = \lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1$$

eq (2) becomes $\frac{ds}{d\theta} = \lim_{\delta\theta \to 0} \frac{\text{chord } PQ}{\delta\theta}$ —— (3)

In ΔPQL

$$(\text{chord } PQ)^2 = (PL)^2 + (QL)^2 \text{ ———— } (4)$$

Again in ΔOPL

$$PL = r \sin \delta\theta$$

and $OL = r \cos \delta\theta$

$$\therefore \; QL = OQ - OL$$

$$= r + \delta r - r\cos\delta\theta$$

Putting values in (4), we have.

$$[\text{chord } PQ]^2 = (r \sin\delta\theta)^2 + (r + \delta r - r\cos\delta\theta)^2$$

$$= \left\{ r\left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots\right]\right\}^2 + \left[r - \delta r - r\left(1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^2}{4!} - \cdots\right)\right]^2$$

$$(\text{chord } PQ)^2 = r^2\delta\theta^2 + \delta r^2$$

From (3)

$$\frac{ds}{d\theta} = \lim_{\delta\theta \to 0} \frac{\text{chord } PQ}{\delta\theta} \text{ ———— } (3)$$

$$\frac{ds}{d\theta} = \lim_{\delta\theta \to 0} \left(\frac{r^2 \delta\theta^2 + \delta r^2}{\cancel{\delta\theta}}\right)^{1/2} \frac{1}{\delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{(r^2 \delta\theta^2 + \delta r^2)^{1/2}}{\delta\theta} = \lim_{\delta\theta \to 0} \left[\frac{r^2}{\delta\theta^2}\delta\theta^2 + \frac{\delta r^2}{\delta\theta^2}\right]^{1/2}$$

$$= \lim_{\delta\theta \to 0} \left[r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2\right]^{1/2}$$

$$= \left[r^2 + \lim_{\delta\theta \to 0}\left(\frac{\delta r}{\delta\theta}\right)^2\right]^{1/2}$$

$$\frac{ds}{d\theta} = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2}$$

Corrollary: If the curve eq is $\theta = f(r)$

$$\frac{ds}{d\theta} = \frac{ds}{dr} \cdot \frac{dr}{d\theta}$$

$$\Rightarrow \frac{ds}{dr} = \frac{ds/d\theta}{dr/d\theta}$$

$$\Rightarrow \frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr}$$

$$= \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} \frac{d\theta}{dr}$$

$$= r^2\left(\frac{d\theta}{dr}\right)^2 + \left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{dr}\right)^2$$

$$\Rightarrow \frac{ds}{dr} = \left(r\frac{d\theta}{dr}\right)^2 + 1$$

$$\Rightarrow \frac{ds}{dr} = \sqrt{\left(r\frac{d\theta}{dr}\right)^2 + 1}$$

Important Formulae

① If the curve eq in the form of parametric i.e.

$$x = f_1(t) \qquad y = f_2(t)$$

Now, $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt}$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \frac{dx}{dt}$$

$$\left[1 + \left(\frac{dy/dt}{dx/dt}\right)^2\right] \frac{dx}{dt}$$

$$\frac{ds}{dt} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2}$$

② Tangent at P makes an angle ψ with x-axis.

$$\text{Slope} = m = \tan\psi = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \tan\psi$$

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\Rightarrow \frac{ds}{dx} = \left[1 + \tan^2\psi\right]^{1/2}$$

$$\Rightarrow \frac{ds}{dx} = \sec\psi$$

$$\Rightarrow \frac{dx}{ds} = \cos\psi$$

(3) $\because \frac{dy}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds}$

$$= \tan\psi \cdot \cos\psi$$

$$\frac{dy}{ds} = \sin\psi$$

$$\psi = \phi + \theta$$

(4) In ΔPLQ

$$\tan \angle PQL = \frac{PL}{QL}$$

$$= \frac{r \sin \delta\theta}{r + \delta r - r\cos\delta\theta}$$

$$= \frac{r\left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots\right]}{r + \delta r - r\left(1 - \frac{(\delta\theta)^2}{2!} + \cdots\right.}$$

$$\tan\angle PQL = \frac{r\,\delta\theta}{\delta r}$$ (Neglecting Higher Powers)

As $Q \to P \Rightarrow \angle PQL \to \phi$
and $\delta\theta, \delta r \to 0$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\cos\phi = \frac{1}{\sec\phi} = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)}} = \frac{dr}{ds} = \frac{dr}{ds}$$

$$\sin\phi = 1 - \cos^2\phi = \frac{\tan\phi}{\cos\phi} = r\frac{d\theta}{ds} \cdot \frac{r\,d\theta}{ds}$$

Evaluate : $\frac{ds}{d\theta}$ for the following curves.

a) $\frac{2a}{r} = 1+\cos\theta$ b) $r = a(1+\cos\theta)$

We know that $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

a) $\frac{2a}{r} = 1+\cos\theta$

$\Rightarrow \cancel{\frac{2a}{1+\cos\theta} = r}$ $\Rightarrow 2a\left(\frac{-1}{r^2}\right)\frac{dr}{d\theta} = -\sin\theta$

$\Rightarrow \cancel{\frac{dr}{d\theta} = 2a(-\sin\theta)}$ $\Rightarrow \frac{dr}{d\theta} = \frac{-r^2\sin\theta}{2a}$

$$\frac{ds}{d\theta} = \left[r^2 + \frac{r^4\sin^2\theta}{4a^2}\right]^{1/2}$$

$$= r^2\left[\frac{1}{r^2} + \frac{\sin^2\theta}{4a^2}\right]^{1/2}$$

$$= r^2\left[\frac{(1+\cos\theta)^2}{4a^2} + \sin^2\theta\right]^{1/2}$$

$$= \frac{r^2}{2a}\left[1+1+2\cos\theta\right]^{1/2}$$

$$= \frac{r^2}{2a}\left[2+2\cos\theta\right]^{1/2}$$ $\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{r^2}{2a}\left[2\times 2\cos^2\frac{\theta}{2}\right]^{1/2}$$

$$\Rightarrow \frac{ds}{d\theta} = \frac{r^2}{a}\cos\left(\frac{\theta}{2}\right)$$

$$\frac{ds}{d\theta} = \frac{4a^2}{(1+\cos\theta)^2}\frac{\sqrt{2}}{2a}(1+\cos\theta)^{1/2}$$

$$\frac{ds}{d\theta} = \frac{2\sqrt{2}a}{(1+\cos\theta)^2}(1+\cos\theta)^{1/2} \Rightarrow \frac{2\sqrt{2}a}{[1+\cos\theta]^{3/2}} = \frac{ds}{d\theta}$$

$$\frac{ds}{d\theta} = 2\sqrt{2}\, a \left(\frac{1}{2\cos^2\frac{\theta}{2}}\right)^{3/2}$$

$$= \frac{2\sqrt{2}\, a}{2\sqrt{2}\,\cos^3\left(\frac{\theta}{2}\right)}$$

$$\frac{ds}{d\theta} = a \sec^3\frac{\theta}{2}$$

b) $r = a(1+\cos\theta)$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$r^2 = a^2(1+\cos\theta)^2$$

$$\frac{ds}{d\theta} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{2a^2 + 2a^2\cos\theta}$$

$$= a\sqrt{2}\,[1+\cos\theta]^{1/2}$$

$$= 2a\cos\left(\frac{\theta}{2}\right)$$

$$\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$$

Q for any curve $y = a \log \sec\left(\frac{x}{a}\right)$

prove that $\frac{ds}{dx} = \sec\left(\frac{x}{a}\right)$ and $\frac{d^2x}{ds^2} = \frac{-1}{2a}\sin\left(\frac{2x}{a}\right)$

$$\frac{ds}{dx} = \pm\sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \frac{1}{\cos\theta} = \frac{-1}{\cos^2\theta}$$

$$\frac{dy}{dx} = a\,\frac{1}{\sec\left(\frac{x}{a}\right)}\left[\sec\frac{x}{a}\tan\frac{x}{a}\right]\left(\frac{1}{a}\right)$$

$$\frac{dy}{dx} = \tan\frac{x}{a}$$

$$\frac{ds}{dx} = \sqrt{1+\tan^2\frac{x}{a}} = \sqrt{\sec^2\frac{x}{a}} = \sec\frac{x}{a}$$

$$\frac{ds}{dx} = \sec\left(\frac{x}{a}\right)$$

$$\frac{dx}{ds} = +\cos\frac{x}{a}$$

$$= -\sin\frac{x}{a}\left(\frac{1}{a}\right)$$

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left(\frac{dx}{ds}\right)$$

$$= \frac{d}{ds}\left(\frac{1}{\sec x/a}\right)$$

$$= \frac{d}{ds}\left(\cos\frac{x}{a}\right)$$

$$= \frac{d}{dx}\left(\cos\frac{x}{a}\right)\frac{dx}{ds}$$

$$= \frac{-1}{a}\sin\frac{x}{a} \quad \cos\frac{x}{a}$$

$$\frac{d^2x}{ds^2} = \frac{-1}{2a}\sin\left(\frac{2x}{a}\right)$$

Ex. For any curve

$$r^m = a^m \cos m\theta$$

T.P. $\frac{ds}{d\theta} = \left(a\left(\sec m\theta\right)^{\frac{m-1}{m}}\right)$

and $a^{2m}\frac{d^2r}{ds^2} + m r^{2m-1} = 0$

Given curve equation $r^m = a^m \cos m\theta$ —— (1)

$\because$ We know that $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ —— (2)

Differentiating (1) w.r.t. θ

$$m r^{m-1}\frac{dr}{d\theta} = -a^m m \sin m\theta$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{a^m \sin m\theta}{r^{m-1}} = -r\tan m\theta \quad \text{—— (a)}$$

Putting values in (2), we have

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{-a^m \sin m\theta}{r^{m-1}}\right)^2}$$

$$= \left[r^2 + \frac{a^{2m}\sin^2 m\theta}{r^{2m-2}}\right]^{1/2}$$

$$= \frac{1}{r^{m-1}}\left[r^{2m} + a^{2m}\sin^2 m\theta\right]^{1/2}$$

$$= \frac{1}{r^{m-1}}\left[a^{2m}\cos^2 m\theta + a^{2m}\sin^2 m\theta\right]^{1/2}$$

$$= \frac{1}{r^{m-1}}\left(a^{2m}\right)^{1/2}$$

$$= \frac{a^m}{\left[a^m \cos m\theta\right]^{\frac{m-1}{m}}}$$

$$\frac{ds}{d\theta} = \frac{a^m}{a^{m-1}(\cos m\theta)^{\frac{m-1}{m}}} = a(\sec m\theta)^{\frac{m-1}{m}} \quad \text{—— (3)}$$

(b) We know $\frac{ds}{dr} = \sqrt{\left(\frac{r\,d\theta}{dr}\right)^2 + 1}$

$$\frac{dr}{ds} = \frac{dr}{d\theta}\cdot\frac{d\theta}{ds} \qquad\qquad \frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right)$$

$$= -\frac{a^m \sin m\theta}{r^{m-1}} \cdot \frac{1}{a(\sec m\theta)^{\frac{m-1}{m}}}$$

$$\frac{dr}{ds} = -\frac{a^{m-1}}{r^{m-1}}\,\frac{\sin m\theta}{(\sec m\theta)^{\frac{m-1}{m}}}$$

$$\frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{-a^{m-1}\sin m\theta}{r^{m-1}\,a(\sec m\theta)^{\frac{m-1}{m}}}\right)$$

$$= -r\tan m\theta \cdot \frac{1}{a(\sec m\theta)^{\frac{m-1}{m}}}$$

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We have

from ②, $\frac{dr}{d\theta} = -r \tan m\theta$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = r\left[1 + \tan^2 m\theta\right]^{1/2}$$

$$= r \sec m\theta$$

$$\frac{dr}{ds} = \frac{dr}{d\theta} \frac{d\theta}{ds}$$

$$= -r \tan m\theta \left(\frac{1}{r \sec m\theta}\right)$$

$$\frac{dr}{ds} = -\sin m\theta$$

$$\frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = -\frac{d}{ds}\left(\sin m\theta\right)$$

$$= -\frac{d}{d\theta}\left(\sin m\theta\right) \frac{d\theta}{ds}$$

$$= -m \cos m\theta \; \frac{1}{r \sec m\theta}$$

$$\frac{d^2r}{ds^2} = -\frac{m}{r} \cos^2 m\theta$$

From $r^m = a^m \cos m\theta$

$$\frac{d^2r}{ds^2} = -\frac{m}{r}\left(\frac{r^m}{a^m}\right)^2$$

$$\frac{d^2r}{ds^2} = \frac{-mr^{2m-1}}{a^{2m}}$$

$$\frac{d^2r}{ds^2} + \frac{mr^{2m-1}}{a^{2m}} = 0$$

$$a^{2m} \frac{d^2r}{ds^2} + mr^{2m-1} = 0$$

Hence Proved.

MY ROUGH NOTE BOOK

Q For the cycloid

$$x = a(1-\cos t)$$
$$y = a(t + \sin t) \quad \text{then} \quad \text{——— (1)}$$

find $\frac{ds}{dt}$, $\frac{ds}{dx}$, $\frac{ds}{dy}$

Given ~~Since~~ $x = a(1-\cos t)$ ——— (1)

$$y = a(t + \sin t)$$

∵ We know that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{——— (2)}$$

$$\frac{dx}{dt} = a\sin t \qquad \frac{dy}{dt} = a(1+\cos t)$$

Put in (2), $\frac{ds}{dt} = \left[a^2\sin^2 t + a^2 + a^2\cos^2 t + 2a^2\cos t\right]^{1/2}$

$$\frac{ds}{dt} = \sqrt{2a^2 + 2a^2\cos t}$$

$1+\cos t = 2\cos^2\frac{t}{2}$

$$= a\sqrt{2}\sqrt{1+\cos t}$$

$$\frac{ds}{dt} = 2a\cos\left(\frac{t}{2}\right) \quad \text{——— (3)}$$

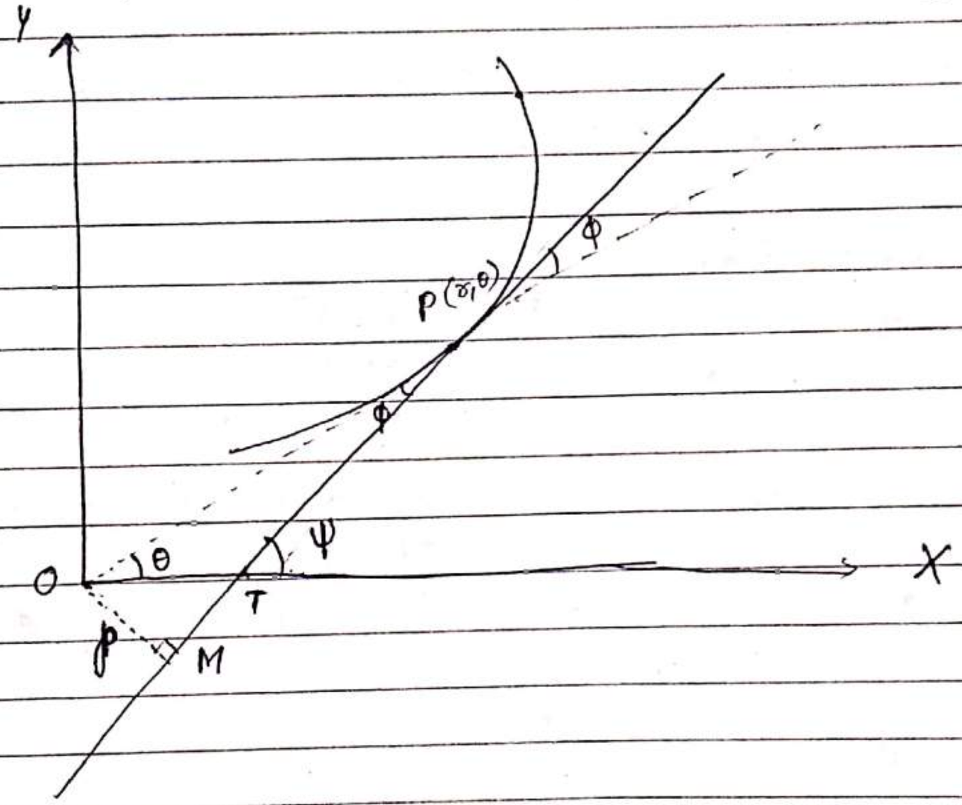
$$\frac{ds}{dx} = \frac{ds}{dt}\cdot\frac{dt}{dx} = \frac{2a\cos(t/2)}{a\sin t} = \operatorname{cosec}\left(\frac{t}{2}\right)$$

$$\frac{ds}{dy} = \frac{ds}{dt}\cdot\frac{dt}{dy} = \frac{2a\cos(t/2)}{a(1+\cos t)} = \sec\left(\frac{t}{2}\right)$$

Angle Between Radius Vector And Tangent.

$$\tan \phi = \frac{r\,d\theta}{dr}$$

Length of perpendicular from pole on the Tangent.



Let curve eq. $r = f(\theta)$ ——— ①

Also that $P(r, \theta)$ be any point on the curve.

Therefore r is the radial vector &

θ is the angle with X- axis or radial vector

Now draw a tangent on P which makes angle ~~ϕ~~ ψ with X axis.

Draw a perpendicular on tangent is OM such that

$$OM = p$$

$$\text{and } \angle OPM = \phi$$

Now, in $\triangle POM$, $p = r \sin \phi$ ——— ②

To find p in form of $r, \theta, \frac{dr}{d\theta}$

$\because$ We know that

$$\tan\phi = r\frac{d\theta}{dr} \quad \text{——— (3)}$$

By eq. (2),

$$p^2 = r^2 \sin^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\Rightarrow \frac{1}{p^2} = \frac{\operatorname{cosec}^2\phi}{r^2} = \frac{1+\cot^2\phi}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{1}{\tan^2\phi}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{1}{\left(r\frac{d\theta}{dr}\right)^2}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{1}{r^2\left(\frac{d\theta}{dr}\right)^2}\right.$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4\left(\frac{d\theta}{dr}\right)^2}$$

$$r = \frac{1}{u} \qquad \Rightarrow \frac{1}{r} = u$$

$$\frac{dr}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta} \qquad \Rightarrow \left(\frac{d\theta}{dr}\right)^2 = u^4\left(\frac{d\theta}{du}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = u^2 + u^4\left(\frac{1}{u^4\left(\frac{d\theta}{du}\right)^2}\right)$$

$$\Rightarrow \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Pedal Equations

If curve eq.^{n} $r = f(\theta)$ can be represent in p and r, then the form is said to be eq of $r = f(\theta)$, where p is the perpendicular length on tangent from the pole.

a) To find the pedal eq. of the curve whose eq. is given in cartesian form:-

Let curve eq. be $f(x,y) = 0$ ——— ①

$\because$ $r^2 = x^2 + y^2$ ——— ②

$\because$ equation of tangent at $f(x,y)$ is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{——— ③}$$

Perpendicular length from origin on ③ is

$$p = \frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \quad \text{——— ④}$$

From ①, ② & ④, eliminate x and y and the resultant eq. is said to be pedal eq. of curve $f(x,y) = 0$.

b) To find pedal equation of the curve whose eq. is given in polar form:-

Let curve eq. be $r = f(\theta)$ ——— ①

$\because$ We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{——— ②}$$

Eliminate θ from ① and ②, then the obtained relationship is known as pedal eq. of curve $r = f(\theta)$.

Alternatively.

$$p = r\sin\phi \quad \text{——— ①}$$

and $$\tan\phi = r\frac{d\theta}{dr} \quad \text{——— ②}$$

Curve eq. $r = f(\theta)$ ——— ③

By ①, ②, ③, eliminate θ & ϕ

then the obtained eq is pedal eq. of the given curve.

Example- Find the pedal equation of the parabola

$$y^2 = 4a(x+a)$$

Given curve eq. $y^2 = 4a(x+a)$ ——— ①

∵ We know that $r^2 = x^2 + y^2$ ——— ②

Differentiating ① w.r.t. x

$$2y \frac{dy}{dx} = 4a$$

$$\boxed{\frac{dy}{dx} = \frac{2a}{y}} \quad \text{——— ③}$$

Therefore eq. of the tangent at (x,y) is given as

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y - y = \frac{2a}{y}(X - x)$$

$$\Rightarrow \frac{2aX}{y} - Y + y - \frac{2ax}{y} = 0 \quad \text{——— ④}$$

Now, Perpendicular distance from origin on ④ is given by

$$p = \frac{y - x\frac{2a}{y}}{\sqrt{1 + \frac{4a^2}{y^2}}} = \frac{y^2 - 2ax}{y} \div \frac{\sqrt{y^2 + 4a^2}}{y}$$

$$p = \frac{y^2 - 2ax}{\sqrt{y^2 + 4a^2}}$$

$$= \frac{2ax + 4a^2}{\sqrt{4ax + 4a^2 + 4a^2}} = \frac{2ax + 4a^2}{4\sqrt{2ax + 2a^2}}$$

$$p = \frac{ax + 2a^2}{\sqrt{ax + 2a^2}} = \sqrt{ax + 2a^2} \quad \text{——— ⑤}$$

By ②, $r^2 = x^2 + y^2 = x^2 + 4ax + 4a^2$.

$$r^2 = (x + 2a)^2 \quad \text{——— ⑥}$$

From (5), (6) $$p^2 = ax + 2a^2$$
$$p^2 = a(x+2a)$$
$$\boxed{p^2 = ar}$$

which is pedal eq. of given curve.

Example - Show that the pedal eq. of the ellipse
$$\frac{l}{r} = 1 + e\cos\theta \qquad (e<1)$$

is $$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right)$$

Given curve eq.
$$\frac{l}{r} = 1 + e\cos\theta \quad \text{———} \star$$

Taking log on both sides, we have
$$\log l - \log r = \log(1+e\cos\theta) \quad \text{———} (1)$$
Differentiating (1) w.r.t. θ
$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+e\cos\theta}(-e\sin\theta)$$
$$\frac{dr}{d\theta} = \frac{r(+e\sin\theta)}{1+e\cos\theta}$$
$$\frac{dr}{d\theta} = \frac{er\sin\theta}{1+e\cos\theta} \quad \text{———} (2)$$

$\because$ We know that
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$
$$= \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 \quad \text{———} (3)$$

$\because$ $$r\frac{d\theta}{dr} = \tan\phi \Rightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}(\cot\phi)^2$$

from (3), $\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r}\frac{dr}{d\theta}\right]$

$$= \frac{1}{r^2}\left[1 + \left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[\frac{(1+e\cos\theta)^2 + e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1+e^2\cos^2\theta + 2e\cos\theta + e^2\sin^2\theta}{\left(\frac{l}{r}\right)^2}\right]$$

$$= \frac{1}{l^2}\left[1+e^2+2e\cos\theta\right]$$

$$= \left[2(1+e\cos\theta)+e^2-1\right]\frac{1}{l^2}$$

$$\frac{1}{p^2} = \left[2\frac{l}{r}+e^2-1\right]\frac{1}{l^2}$$

which is pedal eq. of (4)

Example - Prove that the pedal eq. of the curve ~~ate~~

$$r = a(1-\cos\theta) \quad \text{is}$$

$$p^2 = \frac{r^3}{2a}$$

Given curve eq. is $\quad r = a(1-\cos\theta)$ ——— (1)

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}(a^2\sin^2\theta)$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4}\left(1 - \left(1-\frac{r}{a}\right)^2\right)$$

$$= \frac{1}{r^2} + \frac{a^2}{r^4}\left(\cancel{1} - \cancel{1} - \frac{r^2}{a^2} + \frac{2r}{a}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} - \frac{1}{r^2} + \frac{2r}{a}\frac{a^2}{r^4}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

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$$p^2 = \frac{r^3}{2a}$$

Hence Proved

Ex- Find the angle b/w the radius vector & tangent at a point of the curve $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$\because$ We know that, $\tan\phi = r \frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta}$

$$= \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$\tan\phi = \tan\frac{\theta}{2}$$

$$\boxed{\phi = \frac{\theta}{2}}$$

Ex- Find the pedal eq. of the curve astroid

Given curve eq. $x^{2/3} + y^{2/3} = a^{2/3}$ —— (1)

Parametric form, let $x = a\cos^3 t$

$$y = a\sin^3 t$$

$$y - a\sin^3 t = \frac{dy}{dx}(x - a\cos^3 t)$$

$$\frac{dx}{dt} = -3a\cos^2 t \sin t \qquad \frac{dy}{dt} = 3a\sin^2 t \cos t$$

MY ROUGH NOTE BOOK

$$\frac{dy}{dx} = -\frac{\sin^2 t\cos t}{\cos^2 t\sin t} = -\tan t$$

Eq of tangent,

$$y - a\sin^3 t = -\tan t\,(x - a\cos^3 t) \quad \text{—— (2)}$$

∴ Length of perpendicular from origin

From (2) $0 - a\sin^3 t = -\tan t\,(x - a\cos^3 t$

$$0 = a\cos^3 t\,\tan t - x\tan t + a\sin^3 t$$

Length of perpendicular

$$p = \frac{a\sin^3 t + a\tan t\cos^3 t}{\sqrt{1+\tan^2 t}}$$

$$p = \frac{a\sin^3 t + a\sin t\cos^2 t}{\sec t}$$

$$p = \frac{a\sin t\,[\sin^2 t + \cos^2 t]}{\sec t}$$

$$p = a\sin t\cos t \quad \text{—— (3)}$$

∵ We know $r^2 = x^2 + y^2$

$$r^2 = a^2(\cos^3 t)^2 + a^2(\sin^6 t)$$

$$= a^2[\cos^6 t + \sin^6 t]$$

$$= a^2[\cos^2 t + \sin^2 t][\cos^4 t + \sin^4 t - \cos^2 t\sin^2 t]$$

$$= a^2[\cos^4 t + \sin^4 t - \cos^2 t\sin^2 t]$$

$$= a^2[(\cos^2 t + \sin^2 t)^2 - 3\cos^2 t\sin^2 t]$$

$$= a^2[1 - 3\cos^2 t\sin^2 t]$$

$$r^2 = a^2 - 3p^2$$

which is req. pedal eq. of the given curve.

$(a+b)(a^2+b^2-ab)$
$a^3 + ab^2$
$-a^2b$
$+b^3$
$-ab^2$

Example-1 Find the pedal eq. of the curve

$$a\theta = (r^2 - a^2)^{1/2} - a\cos^{-1}\frac{a}{r}$$

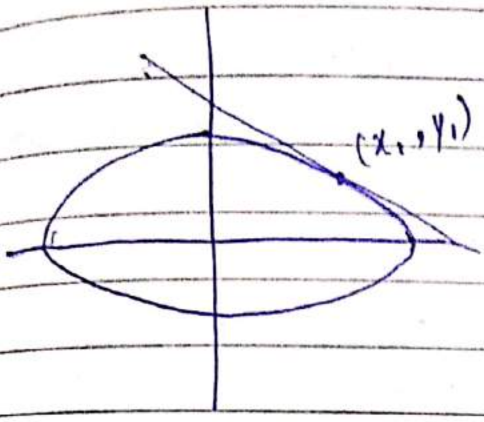
$$\frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{a^2}{r^2 - a^2}\right)\frac{1}{r^2}$$

Q.2 Prove that the pedal eq. of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

Q. Eq. of tangent of an ellipse at (x, y) is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1$$



Eq. of tangent at (x,y) on ellipse is

$$y - y_1 = \frac{dy}{dx}(x - x_1) \quad \text{——} (1)$$

and curve eq.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——} (2)$$

Differentiating (2) w.r.t. x, we have

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}\frac{b^2}{a^2}$$

Putting value in (1),

$$y - y_1 = \frac{-b^2 x}{a^2 y}(x - x_1)$$

$$\frac{y^2}{b^2} - \frac{yy_1}{b^2} = -\frac{x^2}{a^2} + \frac{xx_1}{a^2}$$

$$\Rightarrow 1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} \quad \text{——} (3) \quad (\text{By } (2))$$

$\because$ $(a\cos\theta, b\sin\theta)$ satisfy eq. (2)

$\therefore$ Curve passes through $(a\cos\theta, b\sin\theta)$.

$\therefore$ By (3) eq. tangent passes through $(a\cos\theta, b\sin\theta)$

i.e. $\frac{x \cdot a\cos\theta}{a^2} + \frac{y\, b\sin\theta}{b^2} = 1$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$bx\cos\theta + ay\sin\theta = ab \quad \text{——} (4)$$

Now, length of perpendicular from origin on tangent (4)

$$p = \frac{|-ab|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \frac{1}{p^2} = \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2} \quad \text{——— (5)}$$

Since we know that $r^2 = x^2 + y^2$

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

$$\Rightarrow r^2 = a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta)$$

$$\Rightarrow r^2 = a^2 + b^2 - a\sin^2\theta - b^2\cos^2\theta$$

Put values in eq. (5) $\Rightarrow \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} - \frac{r^2}{a^2b^2}$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2} - \frac{r^2}{a^2b^2}$$

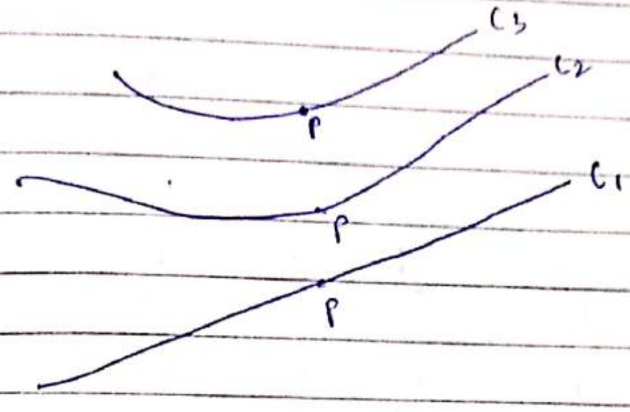
$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

which is pedal eq. of the given curve

Curvature

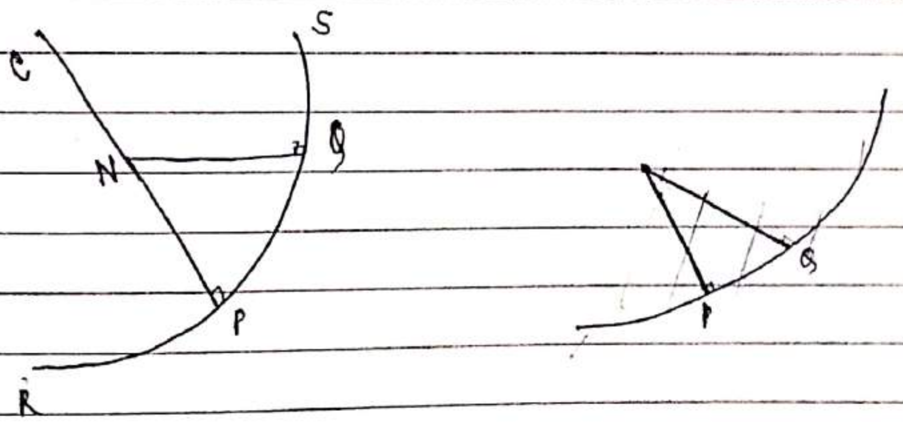
Measurement of the bend in curve is called Curvature In the above figure.

C_1 has curvature 0 but C_2 and C_3 have curvature



Definition

Let a curve RS on which two nearest point P and Q. Now draw normal on P and Q which meets at N.



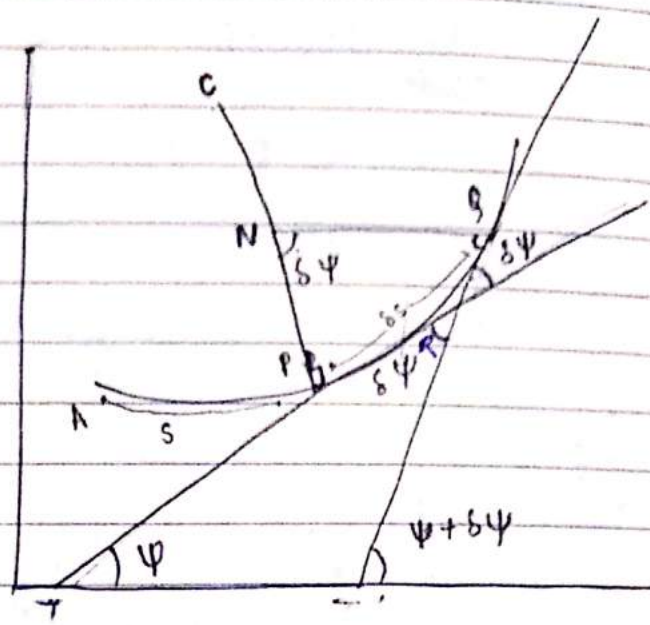
If Q tends to P then N tends to C. In that case C is called centre of curvature and CP is called Radius of Curvature at point P on curve

Reciprocal of distance CP is called ~~Radius of~~ Curvature and Radius of curvature is denoted by ρ.

$$\rho = CP \qquad \text{curvature, } K = \frac{1}{\rho}$$

A circle whose radius is CP then the circle is called circle of Curvature. and a chord from a point P in circle is called CHORD OF CURVATURE.

Formula for Radius of Curvature



In Δ PQN

By sine formula

$$\frac{\sin \angle QNP}{\text{chord } PQ} = \frac{\sin \angle NQP}{PN}$$

$$\frac{\sin \delta\psi}{\sin(\angle NQP)} = \frac{\text{chord } PQ}{PN}$$

$$\Rightarrow \frac{PN}{\text{chord } PQ} = \frac{\sin \angle NQP}{\sin \delta\psi} \quad \text{———— (1)}$$

As $Q \to P$ $\quad \psi \to 0$ and $N \to C$

Therefore radius of curvature

$$\rho = CP$$

$$= \lim_{Q \to P} PN$$

$$= \lim_{\delta\psi \to 0} \frac{\sin \angle NQP}{\sin \delta\psi} \cdot \text{chord } PQ$$

$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin \angle NQP)$$

As $Q \to P$

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$\therefore$ We know that

$$\lim_{Q\to P} \frac{\text{chord } PQ}{\delta s} = 1$$

and as $Q \to P$ $\qquad \angle NBP = \pi/2$

$$\lim_{\delta\psi \to 0} \frac{\sin\psi}{\psi} = 1$$

and $$\lim_{\delta\psi \to 0} \frac{\delta s}{\delta \psi} = \frac{ds}{d\psi}$$

$$\boxed{\rho = \frac{ds}{d\psi}}$$

Curvature, $K = \frac{1}{\rho} = \frac{1}{\frac{ds}{d\psi}} = \frac{d\psi}{ds}$

Cartesian formula for Radius of curvature

$\therefore$ We know that

$$\frac{dy}{dx} = \tan\psi \quad \text{———} \; (i)$$

$$\Rightarrow \quad \frac{d\psi}{ds}$$

Differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\psi)$$

$$= \frac{d}{d\psi} \tan\psi \, \frac{d\psi}{dx}$$

$$= \sec^2\psi \, \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= (1 + \tan^2\psi) \, \frac{1}{\frac{ds}{d\psi}} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}$$

$\because \frac{dy}{dx} = \tan\psi$

$\frac{ds}{d\psi} = \rho$

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right] \frac{1}{\rho} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}} \quad \text{——— (2)}$$

If we replace x by y then (2) becomes

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} \quad \text{——— (3)}$$

Eq. (3) is applicable ~~for~~ when tangent is parallel to y-axis.

Radius of Curvature for Parametric form

Let curve eq. in parametric form is

$$x = x(t) \quad \& \quad y = y(t)$$

Also let $\frac{dx}{dt} = x'$ and $\frac{dy}{dt} = y'$

∴ We know that radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)} \quad \text{——— (1)}$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$= \frac{dy/dt}{dx/dt} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{x'y'' - x''y'}{(x')^2} \cdot \frac{1}{x'} = \frac{x'y'' - y'x''}{x'^3}$$

Putting values in ① we have

$$\rho = \frac{\left[1 + \frac{y'^2}{x'^2}\right]^{3/2}}{\frac{x'y'' - y'x''}{x'^3}}$$

$$\boxed{\rho = \frac{[x'^2 + y'^2]^{3/2}}{x'y'' - y'x''}}$$

Formula for Radius of Curvature when x and y both are form of s.

∵ We know that $\cos\psi = \frac{dx}{ds}$ —— ①

& $\sin\psi = \frac{dy}{ds}$ —— ②

Differentiating ① w.r.t. s of ① and ②, we have

$$\frac{d}{d\psi}(\cos\psi)\frac{d\psi}{ds} = \frac{d^2x}{ds^2}$$

$$= \frac{-1 \sin\psi}{\rho} = \frac{d^2x}{ds^2} \qquad \left(\rho = \frac{ds}{d\psi}\right) —— ③$$

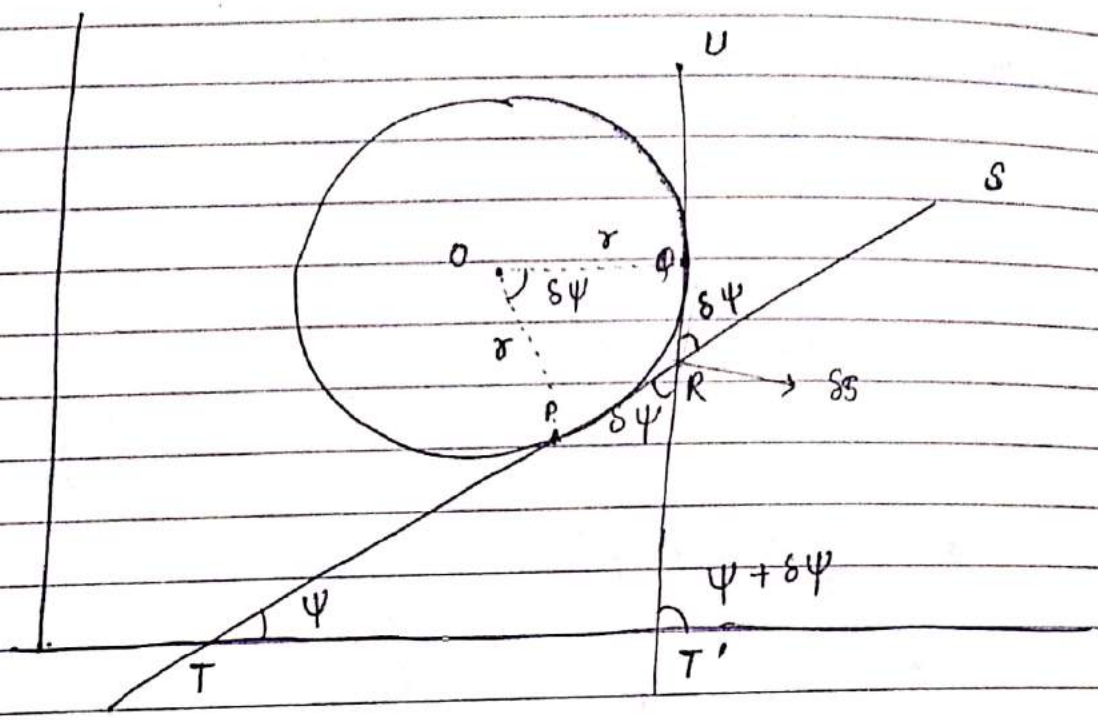
and $\cos\psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2}$

$$\cos\psi \frac{1}{\rho} = \frac{d^2y}{ds^2} —— ④ \qquad \because \rho = \frac{ds}{d\psi}$$

eq. ③^2 + eq. ④^2, we have

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

Curvature of circle :-



$\therefore \angle SRU = \delta\psi$ $\qquad \angle POQ = \angle SRU = \delta\psi$

In Quadrilateral POQR

$$\angle P + \angle Q + \angle O + \angle R = 360^\circ$$

$$\Rightarrow 90 + 90 + \angle O + 180^\circ - \delta\psi = 360^\circ$$

$$\Rightarrow \angle O = \delta\psi$$

$\because$ we know that

$$\frac{\text{arc } PQ}{r} = \angle POQ$$

$$\frac{\delta s}{r} = \delta\psi \qquad \Rightarrow \frac{\delta\psi}{\delta s} = \frac{1}{r}$$

As $Q \to P \quad \Rightarrow \delta s \to 0$

$$\therefore \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{1}{r}$$

$$\frac{d\psi}{ds} = \frac{1}{r}$$

$$\Rightarrow \boxed{\frac{1}{\psi} = \frac{1}{r}}$$

Example Show that for the curve $s = e^{x/c}$

$c\rho = s\sqrt{s^2 - c^2}$

Given curve eq. $s = e^{x/c}$ — ①

$$1 = \frac{d}{ds}\left(e^{x/c}\right)$$

$$= \frac{d}{dx} e^{x/c} \frac{dx}{ds}$$

$$= \frac{1}{c} e^{x/c} \frac{dx}{ds} \qquad \frac{dx}{ds} = \cos\psi$$

$$1 = \frac{1}{c} s \cos\psi$$

$$s = c\sec\psi \quad \text{—} ②$$

Differentiating ② w.r.t. ψ, we have

$$\frac{ds}{d\psi} = c\sec\psi\tan\psi$$

$$\rho = c\,\frac{s}{c}\,\sqrt{\sec^2\psi - 1} \qquad \text{By ② and } \rho = \frac{ds}{d\psi}$$

$$\rho = s\sqrt{\frac{s^2}{c^2} - 1}$$

$$\rho = \frac{s\sqrt{s^2 - c^2}}{c}$$

$$c\rho = s\sqrt{s^2 - c^2}$$

Q Show that the radius of curvature at a pt. is $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3a}{2}\sin 2\theta$

Given curve $x^{2/3} + y^{2/3} = a^{2/3}$

Parametric form of ①,

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

$\because$ We know that radius of curvature

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} \quad \text{——— (2)}$$

when $x' = \frac{dx}{d\theta}$ $\qquad y' = \frac{dy}{d\theta}$

$x'' = \frac{d^2x}{d\theta^2}$ $\qquad y'' = \frac{d^2y}{d\theta^2}$

Now $\frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta)$

$= -3a\sin\theta\cos^2\theta = x'$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta = y'$

$\frac{d^2x}{d\theta^2} = -3a\left[-2\sin^2\theta\cos\theta + \cos^3\theta\right] = x''$

$\frac{d^2y}{d\theta^2} = 3a\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right] = y''$

Putting value in (2), we have,

$$\rho = \frac{\left[9a^2\sin^2\theta\cos^4\theta + 9a^2\sin^4\theta\cos^2\theta\right]^{3/2}}{-3a\sin\theta\cos^2\theta\left[3a(-\sin^3\theta + 2\sin\theta\cos^2\theta)\right] - (-3a)\left[-2\sin^2\theta\cos\theta + \cos^3\right]\left[3a\sin^2\theta\cos\theta\right]}$$

$$= \frac{[3a\sin\theta\cos\theta]^3\left[\cos^2\theta + 1\right]^{3/2}}{9a\sin\theta\cos\theta\left[-\cos\theta(-\sin^3\theta + 2\sin\theta\cos^2\theta)\right] + \sin\theta\left[-2\sin^2\theta\cos\theta + \cos^3\theta\right]}$$

$$= \frac{3[a\sin\theta\cos\theta]^2\left[\cos^2\theta + 1\right]^{3/2}}{\cos\theta\sin\theta\left[-\left(\cancel{-\sin^2\theta + 2\cos^2\theta}\right) + -2\sin^2\theta + \cos\theta\right]}$$

$$\Rightarrow \rho = \frac{\left[9a^2\sin^2\theta\cos^2\theta\right]^{3/2}\left[\sin^2\theta + \cos^2\theta\right]^{3/2}}{-9a^2\sin\theta\cos\theta\left[-\cos\theta\sin^3\theta + 2\sin\theta\cos^3\theta + 2\sin^3\theta\cos\theta - \sin\theta\cos^3\theta\right]}$$

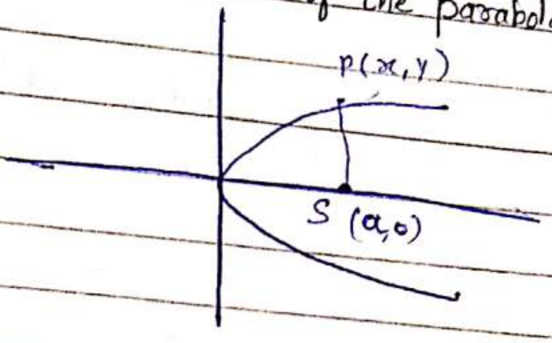
$$\rho = \frac{-3a\sin^2\theta\cos^2\theta}{\sin^3\theta\cos\theta + 2\sin\theta\cos^3\theta - \sin\theta\cos^3\theta} = \frac{-3a\sin\theta\cos\theta}{\sin^2\theta + 2\cos^2\theta - \cos^2\theta}$$

$$\boxed{\rho = \frac{3a}{2}\sin 2\theta}$$

$\because \sin 2\theta = 2\sin\theta\cos\theta$

HoPo

Prove that the radius of curvature at any point P on the parabola $y^2 = 4ax$ is $\frac{2(SP)^{3/2}}{\sqrt{a}}$ where S is the focus of the parabola



$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$2\frac{dy}{dx} = \frac{4a}{y}$$

$$\frac{dy}{dx} = \frac{2a}{y} \quad \text{——} (1)$$

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = \frac{-4a^2}{y^3} \quad \text{——} (2)$$

∵ We know that Radius of Curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1 + \frac{4a^2}{y^2}\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$\rho = \frac{\left[y^2 + 4a^2\right]^{3/2}}{-4a^2} \quad \text{——} (3)$$

$$SP = \sqrt{(x-a)^2 + (y-0)^2}$$
$$= \sqrt{(x-a)^2 + y^2}$$
$$= \sqrt{(x-a)^2 + 4ax} \quad \text{from (1)}$$
$$= \sqrt{(x+a)^2}$$
$$SP = x + a$$

By (3)

$$\rho = \frac{\left[4ax + 4a^2\right]^{3/2}}{4a^2}$$ neglecting -ve sign since radius ≠ -ve

$$\rho = \frac{[4a]^{3/2}\,[x+a]^{3/2}}{4a^2}$$

$$= \frac{8\,a^{3/2}}{4\,a^2}\,(SP)^{3/2}$$

$$\rho = \frac{2\,(SP)^{3/2}}{\sqrt{a}}$$

Q Show that the radius of curvature at any point (x, y) on the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the length of the perpendicular from the origin on the tangent at that pt.

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{——— (1)}$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$-\frac{y^{1/3}}{x^{1/3}} = \frac{dy}{dx} \quad \text{——— (2)}$$

$$+\frac{1}{3}x^{-4/3}y^{1/3} - \frac{1}{3}y^{-2/3}x^{-1/3}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}x^{-4/3}y^{1/3} + \frac{1}{3}y^{-2/3}\frac{1}{x^{1/3}}\frac{y^{1/3}}{x^{1/3}}$$

$$= \frac{1}{3}x^{-4/3}y^{-1/3}\left[y^{2/3} + x^{2/3}\right]$$

$$= \frac{1}{3}x^{-4/3}y^{-1/3}a^{2/3} \quad \text{——— (3)}$$

$\therefore$ radius of curvature, $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1 + \frac{y^{2/3}}{x^{2/3}}\right]^{3/2}}{\frac{1}{3}x^{-4/3}y^{-1/3}a^{2/3}} = \frac{\left[\frac{a^{2/3}}{x^{2/3}}\right]^{3/2}}{\frac{1}{3}x^{-4/3}y^{-1/3}a^{2/3}} \quad \text{by (1)}$$

$$= 3 \frac{a}{x} x^{-4/3} y^{-1/3} a^{2/3}$$

$$= 3 \frac{a^{1/3}}{x^{-1/3} y^{-1/3}}$$

$$\rho = 3 (axy)^{1/3} \quad \text{------} \quad (4)$$

∵ Eq of tangent passes through (x, y) is

$$(Y-y) = \frac{dy}{dx}(X-x)$$

$$(Y-y) = -\left(\frac{y}{x}\right)^{1/3}(X-x)$$

$$Y - y + \left(\frac{y}{x}\right)^{1/3}(X-x) = 0 \quad \text{------} \quad (5)$$

∴ Perpendicular length from origin on (5)

$$p = \frac{\left| -y - x\left(\frac{y}{x}\right)^{1/3} \right|}{\sqrt{\left(\frac{y}{x}\right)^{2/3} + 1}}$$

$$= \frac{y^{2/3} + x^{2/3}}{y^{-1/3}} \div \sqrt{\frac{y^{2/3} + x^{2/3}}{x^{2/3}}}$$

$$= \frac{y^{1/3} a^{2/3}}{\left(\frac{a^{2/3}}{x^{2/3}}\right)^{1/2}} \quad \text{from eq. (1)}$$

$$= y^{1/3} \frac{a^{2/3}}{a^{1/3}} x^{1/3}$$

$$= a^{1/3} x^{1/3} y^{1/3}$$

$$p = (axy)^{1/3} \quad \text{------} \quad (6)$$

From (4) & (6)

$$\rho = 3p$$

Q Show that the curve $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right) + a \tan\psi \sec\psi$ have radius of curvature

$\rho = 2a \sec^3\psi$ and hence show that

$\frac{d^2y}{dx^2} = \frac{1}{2a}$ and also show that D.E. is satisfied by the parabola. $x^2 = 4ay$

$\because$ We know that $\rho = \frac{ds}{d\psi}$ — (1)

Curve eq $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right) + a \tan\psi \sec\psi$ — (2)

$$\frac{ds}{d\psi} = a \frac{\sec^2\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}\left(\frac{1}{2}\right) + a \sec^2\psi \sec\psi + a \tan^2\psi \sec\psi$$

$$\frac{ds}{d\psi} = a\left[\frac{1}{2\sin\left(\frac{\pi}{4} + \frac{\psi}{2}\right)\cos\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}\right] + a \sec\psi\left[\sec^2\psi + \tan^2\psi\right]$$

$$= \frac{a}{\sin\left(\frac{\pi}{2} + \psi\right)} + a \sec\psi\left[\sec^2\psi + \tan^2\psi\right]$$

$$= a \sec\psi\left[1 + \tan^2\psi + \sec^2\psi\right]$$

$$= a \sec\psi\left[\sec^2\psi + \sec^2\psi\right]$$

$$\rho = \frac{ds}{d\psi} = 2a \sec^3\psi \quad \text{— (3)}$$

$\because$ We know that

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = 2a \sec^3\psi \quad [\text{By (3)}]$$

~~Put~~ $\left(\frac{dy}{dx}\right)^2$ $\because \frac{dy}{dx} = \tan\psi$

$$\frac{\left[1 + (\tan\psi)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = 2a\sec^3\psi$$

$$\Rightarrow \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = 2a\sec^3\psi$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2a} \quad \text{——— (4)}$$

Now By eq of parabola

$$x^2 = 4ay$$

$$2x = 4a\frac{dy}{dx}$$

$$\frac{x}{2a} = \frac{dy}{dx}$$

$$\frac{1}{2a} = \frac{d^2y}{dx^2} \quad \text{——— (5)}$$

By (4) and (5), D.E. in eq (4) is satisfies the parabola $x^2 = 4ay$

Ex. find the radius of curvature of curve follium $x^3 + y^3 = 3xya$ at the pt $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ $\quad \rho = \frac{3\sqrt{2}a}{16}$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

∵ Given $x^3 + y^3 = 3axy$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx}$$

$$x^2 - ay = (ax - y^2) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{9a^2/4 - 3a^2/2}{3a^2/2 - 9a^2/4} = -1$$

$$\frac{d^2y}{dx^2} = \frac{\left(2x - a\frac{dy}{dx}\right)(ax - y^2) - \left(a - 2\frac{dy}{dx}\right)(x^2 - ay)}{(ax - y^2)^2}$$

$$= \frac{(3a + a)\left(\frac{3a^2}{\cancel{4}} - \frac{9a^2}{4}\right) - (a + 3a)\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}{\left(\frac{3a^2}{4}\right)^2}$$

MY ROUGH NOTE BOOK

$$= \frac{-8a}{3a^2/4} = \frac{-32}{3a}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \left[1 + (-1)^2\right]^{3/2} \div \left(\frac{-32}{3a}\right)$$

$$= \frac{2\sqrt{2}}{32}(3a)$$

$$\rho = \frac{3\sqrt{2}a}{16}$$ Neglect -ve sign

Ex - If ρ is the radius of the curvature of the curve $y = \frac{ax}{a+x}$ at the pt. (x, y), then prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

$$y = \frac{ax}{a+x}$$

$$\frac{dy}{dx} = \frac{a(a+x) - ax}{(a+x)^2} = \frac{a^2}{(a+x)^2} = \frac{y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}x^2 - 2xy}{x^4} = \frac{y - 2xy}{x^4}$$

$$\rho = \frac{\left[1 + \frac{y^2}{x^4}\right]^{3/2}}{\frac{y-2xy}{x^4}} = \frac{[x^4 + y^2]^{3/2}}{x^6} \div \frac{y-2xy}{x^4}$$

$$= \frac{[x^4+y^2]^{3/2}}{x^2} \times \frac{1}{y-2xy}$$

Formula for Pedal Equation

~~∴ We know that~~

Let pedal eq. of the curve is

$$p = f(r) \quad \text{——} (1)$$

∵ We know that

$$\psi = \theta + \phi \quad \text{——} (2)$$

Differentiating w.r.t. s,

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$\sin\phi = r\frac{d\theta}{ds}$

$$\frac{1}{\rho} = \frac{\sin\phi}{r} + \frac{d\phi}{dr}\,\frac{dr}{ds}$$

$\cos\phi = \frac{dr}{ds}$

$$= \frac{\sin\phi}{r} + \frac{d\phi}{dr}\cdot\cos\phi$$

$p = r\sin\phi$

$$= \frac{1}{r}\left[\underbrace{\sin\phi + r\frac{d\phi}{dr}\cos\phi}\right]$$

$$= \frac{1}{r}\,\frac{d}{dr}\left[r\sin\phi\right]$$

$$\boxed{\frac{1}{\rho} = \frac{1}{r}\,\frac{dp}{dr}}$$

$\because p = r\sin\phi$

$$\Rightarrow \boxed{\rho = r\frac{dr}{dp}}$$

Radius of curvature for polar eqs.

Let curve eq in polar form $r = f(\theta)$ —— (1)

∵ We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{——} (2)$$

Also we know that

$$\rho = r\frac{dr}{dp} \quad \text{——} (3)$$

Differentiate (2) w.r.t. r

$$-2\frac{1}{p^3}\frac{dp}{dr} = \frac{-2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left[\frac{d}{dr}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$-\frac{2}{p^3}\frac{dp}{dr} = \frac{-2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\, 2\frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\cdot\frac{d\theta}{dr}$$

$\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)^2 \frac{d\theta}{dr}$

$$= \frac{1}{r^5}\left[-2r^2 - 4\left(\frac{dr}{d\theta}\right)^2 + 2r\frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\frac{d\theta}{dr}\right]$$

$$\frac{dp}{dr} = \frac{p^3}{r^5}\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\cancel{\frac{dr}{d\theta}}\right]$$

$$\frac{dr}{dp} = \frac{r^5}{p^3}\frac{1}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad \text{——— (4)}$$

From (2)

p_1

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^3} = \left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}$$

Putting value of p in (4)

Now from (3)

$$\rho = r\frac{dr}{dp}$$

$$= \frac{r^6}{p^3}\left[\frac{1}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right]$$

$$= r^6\left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}\left[\frac{1}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right]$$

Here $r^6 = (r^4)^{3/2}$

So,

$$\rho = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}\left[\frac{1}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right]$$

Formula for Radius of Curvature for Tangential Polar Eq.

$\because$ We know that

$$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp} \quad \text{——— (1)}$$

Now, $$\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi}$$

$$\frac{dp}{d\psi} = \frac{dp}{dr}\cos\phi\,\rho \qquad \left[\frac{dr}{ds} = \cos\phi \;\&\; \text{By (1)}\right]$$

$$\frac{dp}{d\psi} = \frac{dp}{dr}\cos\phi\, \frac{r\,dr}{dp} \qquad [\text{by (1)}]$$

$$\Rightarrow \frac{dp}{d\psi} = r\cos\phi \quad \text{——— (2)} \qquad [\text{By Chain Rule}]$$

Also, we know that $p = r\sin\phi$ ——— (3)

Now $(2)^2 + (3)^2$, we have

$$r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2 \quad \text{——— (4)}$$

Differentiating (4) w.r.t. p, we have

$$2r\frac{dr}{dp} = 2p + \frac{d}{dp}\left(\frac{dp}{d\psi}\right)^2$$

$$= 2p + \frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)^2 \frac{d\psi}{dp}$$

$$= 2p + 2\frac{dp}{d\psi}\,\frac{d^2p}{d\psi^2}\,\frac{d\psi}{dp}$$

$$r\frac{dr}{dp} = p + \frac{d^2p}{d\psi^2}$$

$$\Rightarrow \rho = p + \frac{d^2p}{d\psi^2} \qquad \text{from eq (1)}$$

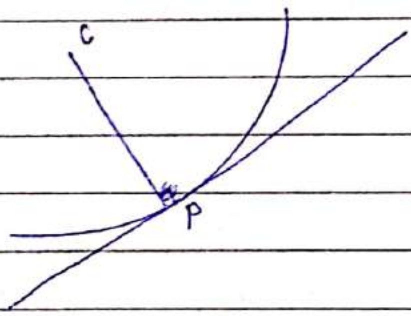
Some Fundamental Definitions

i) Centre of Curvature:

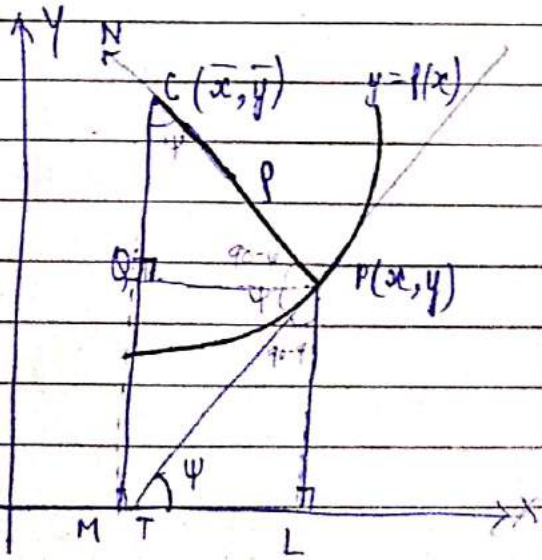
ii) Circle of Curvature:-

iii) chord of curvature: can have infinite no of chord of curvature

iv) Evalute : the locus of centre of curvature of the curve is said to be evalute and the curve is called Invalute of its evalute



Coordinates of Centre of Curvature



Acc to figure, $\bar{x} = OL - ML$

$= OL - PQ$ ——— (1)

and $\bar{y} = MQ + QC$

$= PL + QC$ ——— (2)

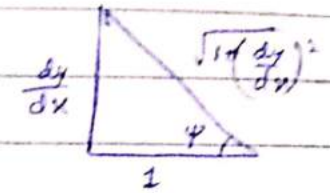
Now in Δ PQC, $PQ = \rho \sin\psi$

and $QC = \rho \cos\psi$

Therefore, $\bar{x} = x - \rho \sin\psi$ ——— (3)

and $\bar{y} = y + \rho \cos\psi$ ——— (4)

$\because$ We know that $\tan\psi = \dfrac{dy}{dx}$



$$\cos\psi = \frac{B}{H} = \frac{1}{\sqrt{1+\tan^2\psi}}$$

$$\sin\psi = \frac{P}{H} = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}}$$

and
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

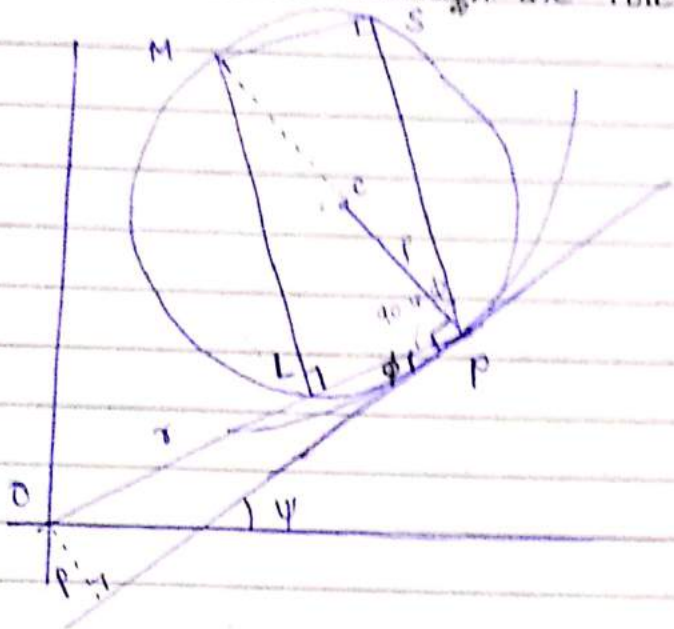
$$\bar{x} = x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \times \frac{dy/dx}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}$$

$$\bar{x} = x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

|||
$$\bar{y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \times \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Length of the chord of the curvature

a) Length of the chord of curvature through the Pole.



Length of the chord passes through Pole

$PL = 2\rho \sin\phi$

$\because$ We know that $p = r\sin\phi$

$$\boxed{PL = 2\rho \frac{p}{r}}$$

b) Length of the chord of curvature perpendicular to radial vector

$SP = 2\rho \cos\phi$

$\because$ $p = r\sin\phi$

$= \cancel{r\sqrt{1-\cos\phi}}$ $\quad \cos\phi = \sqrt{1-\sin^2\phi}$

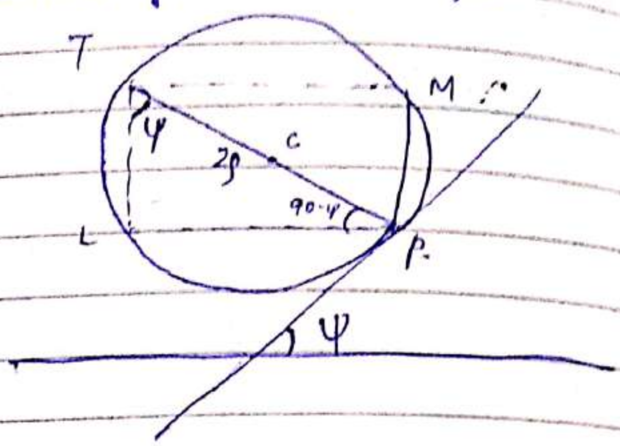
$= \sqrt{1-\left(\frac{p}{r}\right)^2}$

$$\boxed{PS = \frac{2\rho}{r}\sqrt{r^2-p^2}}$$

c) Length of the chord of curvature parallel to the Axes

$PL = 2\rho \sin\psi$

$PM = LT \quad 2\rho \cos\psi$

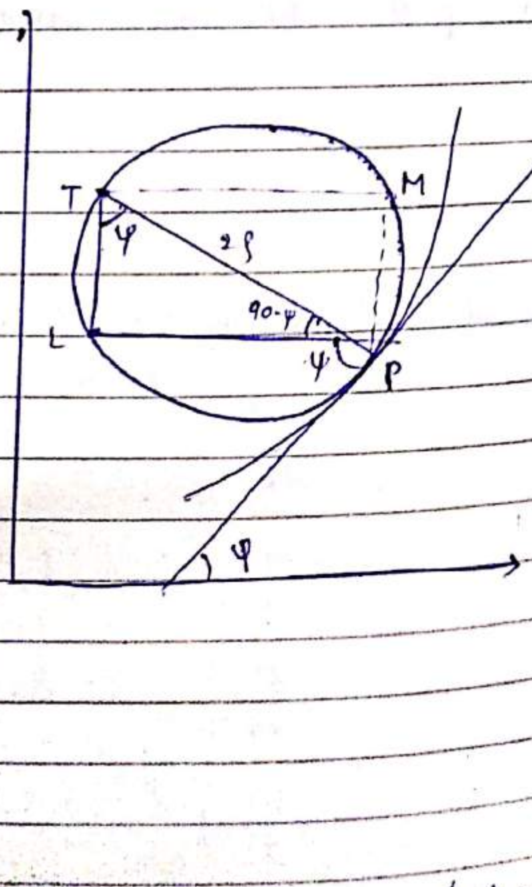


Ex - Prove that the chord of curvature parallel to y axis of the curve $y = a\log\left[\sec\left(\frac{x}{a}\right)\right]$ is of constant length.

Given curve $y = a\log\left[\sec\left(\frac{x}{a}\right)\right]$ —— ①

∵ We know that length of the chord parallel to y axis i.e.

$PM = 2\rho\cos\psi$ —— ②



Also we know that radius of curvature

$$\rho = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2} \quad \text{——} ③$$

$$\frac{dy}{dx} = \frac{a}{\sec\left(\frac{x}{a}\right)} \sec(x/a)\tan\left(\frac{x}{a}\right)\frac{1}{a}$$

$$\frac{dy}{dx} = \tan\left(\frac{x}{a}\right) \quad \text{——} ④$$

Again differentiate ④ w.r.t. x.

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\left(\frac{x}{a}\right) \quad \text{——} ⑤$$

Putting value in ③

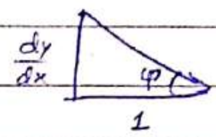
$$\rho = \left[1+\tan^2\left(\frac{x}{a}\right)\right]^{3/2} \div \frac{1}{a}\sec^2\left(\frac{x}{a}\right) \Rightarrow a\sec\left(\frac{x}{a}\right) = \rho \text{ —⑥}$$

$\because$ We know that $\tan\psi = \frac{dy}{dx}$

from eq. (4) $\tan\psi = \tan\left(\frac{x}{a}\right)$

$$\Rightarrow \psi = \frac{x}{a}$$

$$\therefore \cos\psi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$



$\tan\varphi = \frac{\left(\frac{dy}{dx}\right)}{1}$

$$= \frac{1}{\sqrt{1+\tan^2(\psi)}}$$

$$= \frac{1}{\sqrt{1+\tan^2\left(\frac{x}{a}\right)}}$$

$$\cos\psi = \frac{1}{\sec x/a} = \cos\frac{x}{a} \longleftarrow (7)$$

Putting value in (2) by (6), (7)

Length of the ~~PM~~ chord parallel to y axis $= 2\rho\cos\psi$

$$= 2a\sec\left(\frac{x}{a}\right)\frac{1}{\sec\left(\frac{x}{a}\right)}$$

$$= 2a$$

<u>Example</u> $\rightarrow$ Find the eq. of circle of curvature of the curve ~~to~~

$y = x^3 + 2x^2 + x + 1$ —(1) for the point $(0,1)$

$$\frac{dy}{dx} = 3x^2 + 4x + 1$$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

$$\left(\frac{d^2y}{dx^2}\right) = 6x + 4$$

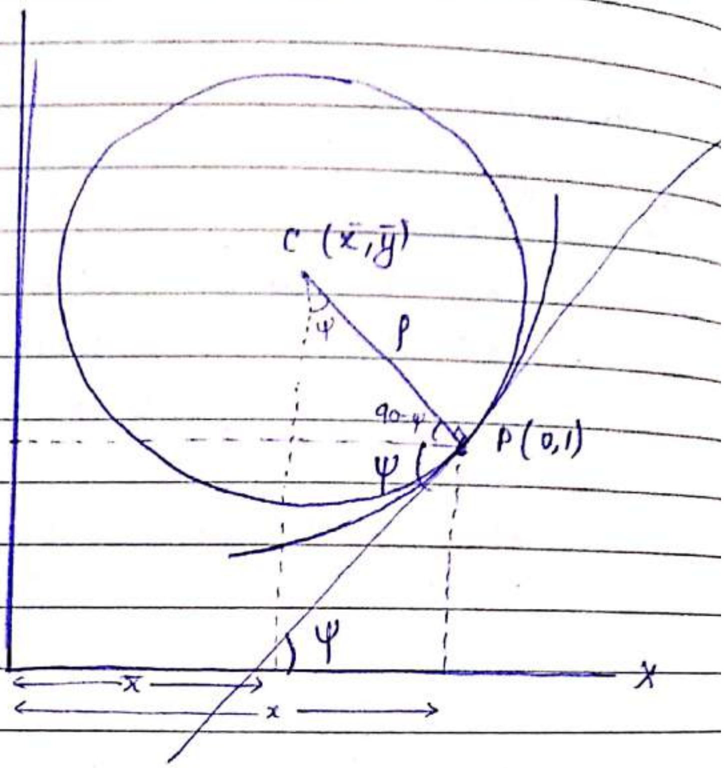
$$\left(\frac{d^2y}{dx^2}\right)_{(0,1)} = 4$$

∵ we know that Radius of curvature

$$\rho = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2}$$

$$= \frac{2\sqrt{2}}{4}$$

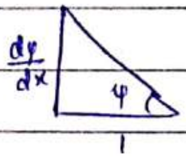
$$\rho = \frac{1}{\sqrt{2}}$$



Also we know that

$$\bar{x} = x - \rho \sin \psi$$

$$\because \tan\psi = \frac{dy}{dx}$$



$$\sin\psi = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}}$$

$= \cancel{cscn}$.

$$\bar{x} = 0 - \frac{1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2}$$

|||

$$\bar{y} = y + \rho\cos\psi$$

$$\bar{y} = 1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{3}{2}$$

∴ Eq of circle of curvature is

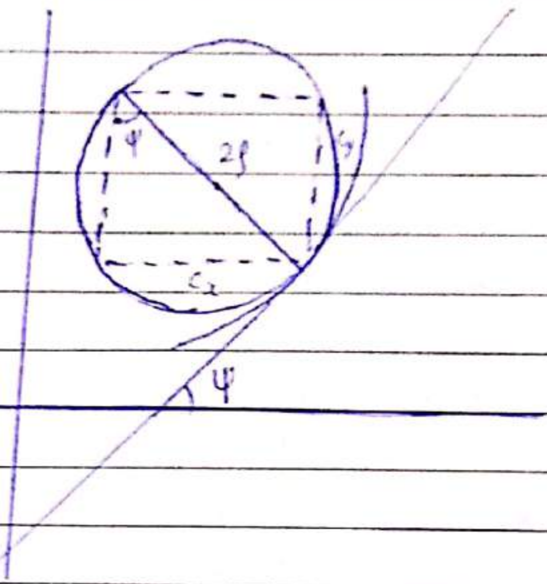
$$\left[x - \left(\frac{-1}{2}\right)\right]^2 + \left(y - \frac{3}{2}\right)^2 = \rho^2$$

$$x^2 + \frac{1}{4} + x + y^2 + \frac{9}{4} - 3y = \frac{1}{2}$$

$$x^2 + y^2 + 2x - 3y + 2 = 0$$

Ex. If C_x and C_y be the chords of curvature parallel to the axis respectively at any point of the curve $y = ae^{x/a}$, then prove that

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}$$



Given curve eq.

$$y = ae^{x/a} \quad \text{——— (1)}$$

Acc. to diagram,

$$C_x = 2\rho \sin\psi \quad \text{——— (2)}$$

$$C_y = 2\rho \cos\psi = \text{length of chord} \parallel \text{to y axis} \quad \text{——— (3)}$$

We know radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} \quad \text{——— (4)}$$

$$\frac{dy}{dx} = a\, e^{x/a} \times \frac{1}{a} = e^{x/a}$$

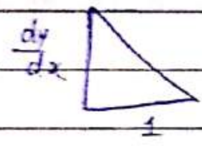
III

$$\frac{d^2y}{dx^2} = \frac{1}{a} e^{x/a}$$

By (4),

$$\rho = a\frac{\left[1 + e^{2x/a}\right]^{3/2}}{e^{x/a}}$$

Now we know that



$$\tan\psi = \frac{dy}{dx}$$

$$\sin\psi = \frac{dy/dx}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\cos\psi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

By (2) and (3)

$$C_x = 2\rho \sin\psi$$
$$= 2a \frac{[1+e^{2x/a}]^{3/2}}{e^{x/a}} \times \frac{e^{x/a}}{\sqrt{1+e^{2x/a}}}$$
$$= 2a\left(1+e^{2x/a}\right) \quad \text{———} (5)$$

$$C_y = 2\rho \cos\psi$$
$$= 2a \frac{[1+e^{2x/a}]^{3/2}}{e^{x/a}} \times \frac{1}{\sqrt{1+e^{2x/a}}}$$
$$= \frac{2a[1+e^{2x/a}]}{e^{x/a}}$$

LHS.
$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{4a^2(1+e^{2x/a})^2} + \frac{e^{2x/a}}{4a^2[1+e^{2x/a}]^2}$$
$$= \frac{1}{4a(1+e^{2x/a})}$$
$$= \frac{1}{2a[2a(1+e^{2x/a})]}$$
$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2a\,C_x} \quad \text{from (5)}$$

Ex. Show that for the Cardioid $r = a(1+\cos\theta)$

a) $\rho \propto \sqrt{r}$

b) $9(\rho_1^2 + \rho_2^2) = 16a^2$

where ρ_1 & ρ_2 are the Radius of Curvature of the extrimities of any chord which passes through the ~~chord~~ pole.

$r = a(1+\cos\theta)$

$\because$ We know that

$$\rho = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} \left[\frac{1}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right] \quad \text{———} (1)$$

$$\frac{dr}{d\theta} = -a\sin\theta \quad \text{———} (2)$$

$$a^2\sin^2\theta = a^2 - (r-a)^2 = 2ar - r^2 \quad \text{———} (3)$$

$$\frac{d^2r}{d\theta^2} = -a\cos\theta = a-r \quad \text{——— (4)}$$

Put above values in (1)

$$\rho = \left[r^2 + a^2\sin^2\theta\right]^{3/2}\left[\frac{1}{r^2 + 2a^2\sin^2\theta - r(a-r)}\right]$$

from (3),

$$\rho = \frac{\left[r^2+2ar-r^2\right]^{3/2}}{r^2+4ar-2r^2+r^2-ar}$$

$$= \frac{\left[2ar\right]^{3/2}}{3ar}$$

$$\rho = \frac{\sqrt{8}}{9}(ar)^{\frac{3}{2}-1} = \frac{\sqrt{8}}{9}\sqrt{ar} \quad \text{——— (5)}$$

$$\rho \propto \sqrt{r}$$

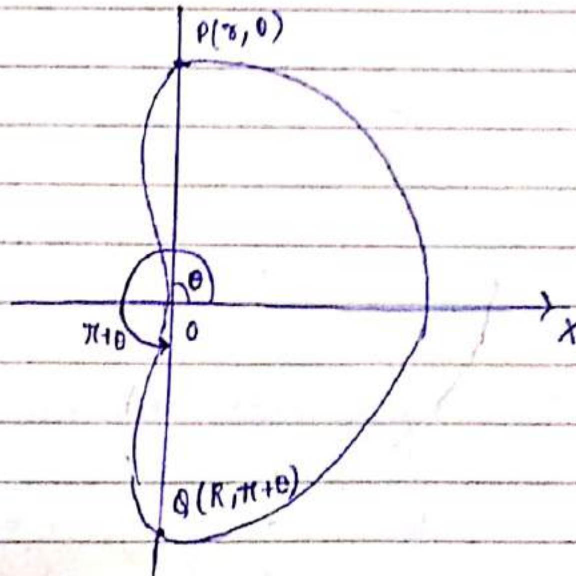
ii) Let O be the pole and OX be the initial line

Let PQ be the chord of the curve passing through the pole.

If $P(r,\theta)$ then $Q(R, \pi+\theta)$

~~we know that~~

From eq. (5)



$$\rho = \frac{2^{3/2}\sqrt{ar}}{3}$$

$$\rho^2 = \frac{8}{9}ar$$

At $P(r,\theta)$ $\quad \rho_1^2 = \frac{8}{9}ar$

At $Q(R,\pi+\theta)$ $\quad \rho_2^2 = \frac{8}{9}aR$

Now $\rho_1^2 + \rho_2^2 = \frac{8a}{9}\left[r+R\right]$

$$= \frac{8a}{9}\left[a(1+\cos\theta) + a(1+\cos(\pi+\theta))\right]$$

$$= \frac{8a}{9}\left[a(1+\cos\theta + 1-\cos\theta)\right]$$

$$= \frac{8a}{9} \times 2a$$

$$\Rightarrow \rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$$

$$\Rightarrow 9\left[\rho_1^2 + \rho_2^2\right] = 16a^2$$

Hence Proved.

Alternative

Given curve eq.

$$r = a(1+\cos\theta)$$

Taking log on both sides

$$\log r = \log a(1+\cos\theta) \quad \text{—— (1)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan\left(\frac{\theta}{2}\right) \qquad \tan\phi = r\frac{d\theta}{dr}$$

$$1/\tan\phi = -\tan\left(\frac{\theta}{2}\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2} \quad \text{—— (2)}$$

$\because$ we know that $p = r\sin\phi$

$$p = r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$p = r\cos\frac{\theta}{2}$$

$$r = a(1+\cos\theta) = 2a\cos^2\frac{\theta}{2} = 2a\frac{p^2}{r^2}$$

Curve Eq. $\Rightarrow r^3 = 2ap^2$ —— (3)

$\because$ We know that

$$\rho = r\frac{dr}{dp} \qquad\qquad r^3 = 2ap^2$$

From (3) $\quad r^3 = 2ap^2$

$$3r^2\frac{dr}{dp} = 4ap$$

$$r\frac{dr}{dp} = \frac{4ap}{3r}$$

$$= \frac{4ap}{3r}$$

$$= 4a\sqrt{\frac{r^3}{2a}}\left(\frac{1}{3r}\right)$$

$$\rho^2 = \frac{16a^2}{9r^2}\frac{r^3}{2a}$$

$$\rho^2 = \frac{8ar}{9} \quad \text{——— (4)}$$

$$\Rightarrow \quad \rho = \frac{\sqrt{8ar}}{3} \qquad \because \frac{\sqrt{8a}}{3} = \text{constant}$$

$$\Rightarrow \quad \rho \propto \sqrt{r} \quad \text{——— (5)}$$

Curvature at the Origin

$\because$ We know that Radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

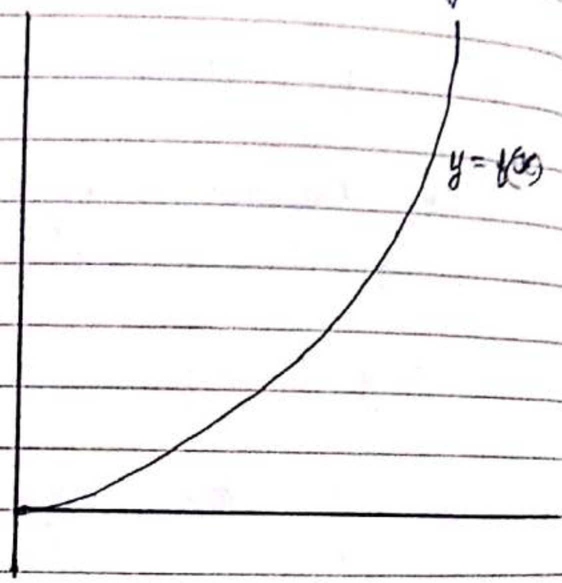
Let $\quad p = \left.\frac{dy}{dx}\right|_{(0,0)}$

$$q = \left.\frac{d^2y}{dx^2}\right|_{(0,0)}$$

$$\therefore \quad \boxed{\rho = \frac{[1+p^2]^{3/2}}{q}}$$

NEWTON'S METHOD

Case I: If curve passes through origin and x axis is tangent on curve at origin.



$$p = \left.\frac{dy}{dx}\right|_{(0,0)} = \text{slope at origin} = 0$$

as $\tan 0 = \tan 0° = 0$

$$\rho = \frac{[1 + p^2]^{3/2}}{q}$$

$$\rho = \frac{1}{q} \quad \text{—— (1)}$$

Now by Maclaurin's theorem

$$y = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \quad \text{—— (2)}$$

$y = f(x+0)$
$y = f(x)$

$\because$ Curve passes through origin and x axis is tangent at origin.

$\therefore \quad f(0) = y(0) = 0$

and $\quad f'(0) = 0$

$\therefore$ eq. (2) becomes

$$y = \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \quad \text{—— (3)}$$

Multiplying $\frac{2}{x^2}$ in eq. (3) and taking limit $x \to 0$ & also take limit $y \to 0$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2y}{x^2} = f''(0) = q$$

By (1) $\quad \therefore \rho = \frac{1}{\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{2y}{x^2}} \Rightarrow \boxed{\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}}$

Case II : If curve passes through origin and y axis is tangent on curve at origin

$$p = \frac{dy}{dx}\Big|_{(0,0)} = \text{slope at origin} = \infty$$

$$\because \tan 90^\circ = \infty$$

$$\rho = \frac{[1+p^2]^{3/2}}{q} = \frac{p^3\left[\frac{1}{p^2}+1\right]^{3/2}}{q} = \frac{p^3}{q} \quad \text{——— (1)}$$

By Maclaurin's expansion

$$y = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$y = xp + \frac{x^2}{2} q + \frac{x^3}{3!} f'''(0) + \ldots \quad \text{——— (2)}$$

Multiplying eq. (2) by $\frac{1}{x}$ and taking lim $x \to 0$ & $y \to 0$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y}{x} = p + \lim_{x\to 0}\left[\frac{x}{2} q + \frac{x^2}{3!} f'''(0) + \ldots\right]$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y}{x} = p \quad \text{——— (3)}$$

Multiplying eq. (2) by $\frac{2}{x^2}$ we get (and also take lim $x\to 0$ & $y \to 0$)

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{2y}{x^2} = \lim_{x\to 0} \frac{2p}{x} + q + \lim_{x\to 0} \frac{2}{x^2}\left[\frac{x^3}{3!} f'''(0) + \ldots\right]$$

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{2y}{x^2} = q \quad \text{——— (4)}$$

From (1), (3) and (4),

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{y}{x}\right)^3 \div \left(\frac{2y}{x^2}\right)$$

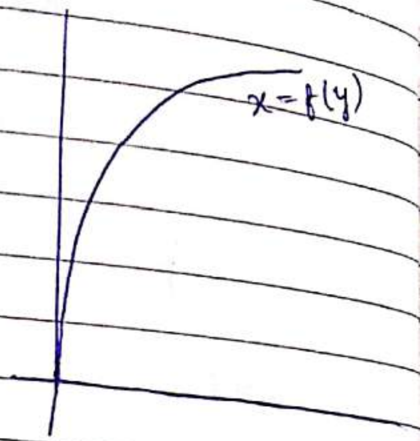
$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^3}{x^3} \times \frac{x^2}{2y}$$

$$\boxed{\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}}$$

Alternatively

$$\left.\frac{dy}{dx}\right|_{0,0} = \infty$$

$$\left.\frac{dx}{dy}\right|_{0,0} = 0$$



$\because$ We know that

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}$$

By Maclaurin's expansion,

$$x = f(y+0)$$

$$x = f(0) + y f'(0) + \frac{y^2}{2!} f''(0) + \frac{y^3}{3!} f'''(0) + \ldots$$

$$f'(0) = \left.\frac{dx}{dy}\right|_{0,0}$$

$$x = 0 + 0 + \frac{y^2}{2!} f''(0) + \frac{y^3}{3!} f'''(0) + \ldots$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2x}{y^2} = \lim_{\substack{x\to 0\\ y\to 0}} f''(0)$$

$$\rho = \frac{1}{\lim_{\substack{x\to 0\\ y\to 0}} \frac{2x}{y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

$$\boxed{\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}}$$

Case III : If curve eq. in polar form $r = f(\theta)$

$$\because \rho = \lim_{x,y \to 0} \frac{x^2}{2y}$$

put $x = r\cos\theta$ $\quad y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\rho = \lim_{r,\theta \to 0} \frac{r^2\cos^2\theta}{2 r\sin\theta}$$

$$\cos^2\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)^2$$

$$\sin\theta \quad \left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

$$\rho = \lim_{r,\theta \to 0} \frac{r^2\left(1 - \frac{\theta^2}{2!}\right)^2}{2 r\left(\theta - \frac{\theta^3}{3!}\right)} = \lim_{\substack{r \to 0,\\ \theta \to 0}} \frac{r^2\left(1 + \frac{\theta^4}{4} - 2\theta^2\right)}{2 r\left(\theta - \frac{\theta^3}{6}\right)}$$

$$\rho = \lim_{r,\theta \to 0} \frac{r}{2\theta} \quad \text{——(1)}$$

By (1) radius of curvature can be find out when curve passes through pole and initial line at pole is tangent.

\# $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Shifting to origin $x = X + h \Rightarrow X = x - h$, $h - h = 0$

$y = Y + k$ $\quad Y = y - k = k - k = 0$ $\quad , h, K$

Ex- Apply Newton's Method to prove that the radius of curvature at the lowest point of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ is equal to c.

∴ We know that lowest point on catenary $y = c\cosh\left(\frac{x}{2}\right)$ is $(0,c)$

By using shifting of origin

put $x = X + 0$

$y = Y + c$

we have

$$Y + c = c\cosh\left(\frac{X}{c}\right) \quad \text{--- (1)}$$

Here eq. (1) passes through origin

$$\cosh\left(\frac{x}{2a}\right) = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \dots + 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \dots}{2}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

So,

$$Y + c = c\cosh\left(\frac{X}{c}\right)$$

$$Y + \cancel{c} = c\left(\cancel{1} + \left(\frac{X}{c}\right)^2 \frac{1}{2!} + \left(\frac{X}{c}\right)^4 \frac{1}{4!} + \dots\right)$$

$$Y = c\left(\frac{X^2}{c^2\,2!} + \left(\frac{X}{c}\right)^4 \frac{1}{4!} + \dots\right) \quad \text{--- (2)}$$

For tangent put lowest degree terms equal to zero in eq. (2), we have

$$Y = 0$$

Therefore X axis is tangent on curve --- (2)

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$$

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2c\left[\frac{x^2}{2!c^2} + \frac{x^4}{4!c^4} + \cdots\right]}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1}{2c\left(\frac{1}{2c^2} + \frac{x^4}{4!c^4} + \cdots\right]}$$

$$\rho = c$$

$\because$ We know that acc. to shifting of origin radius of curvature does not change. So,

radius of curvature, $\rho = c$

Ex. Find the Radius of Curvature at the Origin of the Curve $5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0$

Clearly the given curve passes through Origin, So, its tangent $x = 0$ (lowest degree term)

so tangent is y axis

Since we know that

Radius of curvature $\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$

so divide given curve eq. by $2x$ and put limit $\substack{x \to 0 \\ y \to 0}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{5x^2}{2} + \frac{7 \cdot y^3}{2x} + 2xy + x + \frac{3}{2}y + 2 + (x+1)\frac{y^2}{2x} = 0$$

$y \cdot \rho$

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} = -2$$

Since ρ can't be negative

$$\boxed{\rho = 2}$$

Ex. Cardioid $r = a(1-\cos\theta)$

T.P. $\rho \big|_{pole} = 0$

$r = 0$

$a(1-\cos\theta) = 0$

$\cos\theta = 1$

$\cos\theta = 2n\pi \qquad n = 0, 1, 2$

$\theta = 0, 2\pi, 4\pi, \ldots$

$$\rho = \lim_{\theta, r \to 0} \frac{r}{2\theta} = \lim_{r, \theta \to 0} \frac{a}{2} \frac{(1-\cos\theta)}{\theta}$$

$$= \frac{a}{2}(0)$$

$$= 0$$

L - Hospital Rule

$$\lim_{t \to a} \frac{f(t)}{g(t)} \qquad \frac{\infty}{\infty} \text{ or } \frac{0}{0}$$

$$= \lim_{t \to a} \frac{f'(t)}{g'(t)} \qquad \frac{\infty}{\infty} \text{ or } \frac{0}{0}$$

$$= \lim_{t \to 0} \frac{f''(t)}{g''(t)}$$

Q. $\lim_{t \to \infty} \frac{t^n}{e^t} \qquad \frac{\infty}{\infty}$

$$= \lim_{t \to \infty} \frac{n t^{n-1}}{e^t} \qquad \frac{\infty}{\infty}$$

$$= \lim_{t \to \infty} \frac{n(n-1)\, t^{n-2}}{e^t} \qquad \frac{\infty}{\infty}$$

$$= \lim_{t \to \infty} \frac{n!\, t^0}{e^t} = \frac{n!}{e^t}$$

$$= \frac{\text{finite}}{\infty} = 0$$

Advanced Integral Calculus

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Gamma Function

$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$ Euler's second Integral

a) $\Gamma n = n-1\, \Gamma n-1$
b) if $n \in N$ then $\Gamma n = (n-1)!$
c) $\Gamma 1/2 = \sqrt{\pi}$ $\Gamma 0 = \infty$ & $\Gamma 1-n = \infty$ for $n \in N$
$\Gamma 1 = 1$

Here $\Gamma n = (n-1)\,\Gamma n-1$
$= (n-1)(n-2)\,\Gamma n-2$
$= (n-1)(n-2)(n-3)\ldots 2.1\,\Gamma 1^{=1}$
$= (n-1)(n-2)(n-3)\ldots 3.2.1$
$= (n-1)!$

Properties of Gamma Function

Prop 1. Prove that
a) $\Gamma n+1 = n\,\Gamma n$ $n>0$ Recurrence Formula
b) $\Gamma 1 = 1$
c) $\Gamma n+1 = n!$ Relation b/w Gamma Function & factorial

a) We know that
$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$ —①
$\Gamma n+1 = \int_0^\infty e^{-t}\, t^{n+1-1}\, dt$
$= \int_0^\infty t^{n+\cancel{1}-\cancel{1}}\, e^{-t} dt$
By ILATE
$= \left[-t^n e^{-t}\right]_0^\infty - \int_0^\infty (+t^n e^{-t})\, dt$
By L Hospital Rule
$= \left[\frac{n!}{e^t}\right]_0^\infty - \int_0^\infty n t^{n-1}(-e^{-t})\, dt$
$= 0 + n\int_0^\infty t^{n-1} e^{-t}\, dt$ from ①
$\Gamma n+1 = n\,\Gamma n$

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b) $\because$ we know that

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$\Gamma 1 = \int_0^\infty e^{-t}\, t^{0}\, dt$$

$$= -\left[e^{-t}\right]_0^\infty$$

$$= -\left[\frac{1}{e^t}\right]_0^\infty$$

$$= -\left[e^{-\infty} - e^{-0}\right]$$

$$= -\left[0-1\right]$$

$$= 1$$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

c) $\Gamma n+1 = n!$

$\because \Gamma n+1 = n\,\Gamma n$

Also $\Gamma n = (n-1)!$

$$\Gamma n+1 = n\,(n-1)!$$

$$= n\,(n-1)(n-2)(n-3) \ldots 2 \cdot 1$$

$$= n!$$

Transformation of Gamma function

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt \quad \text{—— (1)}$$

[A] Put $t = \log\left(\frac{1}{s}\right)$ in eq. (1)

$$e^t = \frac{1}{s} \Rightarrow s = e^{-t}$$

$$\Rightarrow ds = -e^{-t}\, dt$$

as $t \to 0$ $s \to 1$

$t \to \infty$ $s \to 0$

$\therefore$ Eq. (1) becomes

$$\Gamma n = -\int_1^0 \left[\log\left(\frac{1}{s}\right)\right]^{n-1} ds$$

$$\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

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$$\Rightarrow \quad \Gamma n = \int_0^1 \left[\log\left(\frac{1}{s}\right)\right]^{n-1} ds$$

[B] Put $t^n = s$ in eq. ①

$n t^{n-1} \, dt = ds$

as $t \to 0$ $\quad s \to 0$

$t \to \infty$ $\quad s \to \infty$

$$\Gamma n = \int_0^\infty e^{-s^{1/n}} \frac{1}{n} ds$$

$$\Gamma n = \frac{1}{n} \int_0^\infty e^{-s^{1/n}} ds$$

$$n \Gamma n = \int_0^\infty e^{-s^{1/n}} ds$$

$$\Gamma(n+1) = \int_0^\infty e^{-s^{1/n}} ds$$

[C] Put $t = az$ in eq. ①

$dt = a\,dz$

as $t \to 0$ $\quad z \to 0$

as $t \to \infty$ $\quad z \to \infty$

$\therefore$ Eq. ① becomes

$$\Gamma n = \int_0^\infty e^{-az} (az)^{n-1} a\,dz$$

$$\Gamma n = \int_0^\infty a^n e^{-az} z^{n-1} dz$$

Replace z by t

$$\Gamma n = a^n \int_0^\infty e^{-at} t^{n-1} dt$$

$$\frac{\Gamma n}{a^n} = \int_0^\infty e^{-at} t^{n-1} dt$$

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[D] Put $t = s^2$ in eq. ①

$dt = 2s\, ds$

at $t \to 0 \Rightarrow s \to 0$

$t \to \infty \Rightarrow s \to \infty$

$$\Gamma n = \int_0^\infty e^{-s^2} s^{2(n-1)} 2s\, ds$$

$$\Gamma n = 2\int_0^\infty e^{-s^2} s^{2n} s^{-1}\, ds$$

$$= 2\int_0^\infty e^{-s^2} s^{2n-1}\, ds$$

Replace s by t or put $s = t$

$$\Gamma n = 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$$

To Prove

$$\Gamma 1/2 = \sqrt{\pi}$$

$\because$ $n\Gamma n = \Gamma n+1 = \int_0^\infty e^{-s^{1/n}}\, ds$ —— ①

Put $n = 1/2$

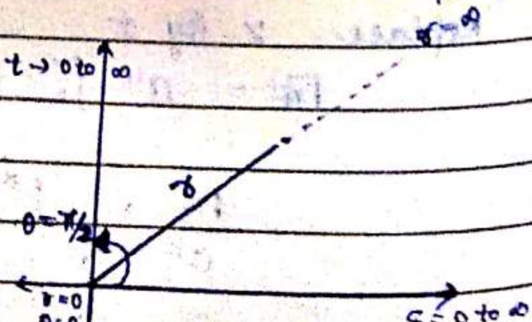
$\frac{1}{2}\Gamma 1/2 = \int_0^\infty e^{-s^2}\, ds$ —— ②

III $\frac{1}{2}\Gamma 1/2 = \int_0^\infty e^{-t^2}\, dt$ —— ③

Multiply ② by ③

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-s^2} e^{-t^2}\, ds\, dt$$

$$(\Gamma 1/2)^2 = 4\int_0^\infty\int_0^\infty e^{-(s^2+t^2)}\, ds\, dt$$



S, $\int_0^\infty\int_0^\infty ds\, dt$

for θ $\int_0^{\pi/2}$

for r $\int_0^\infty$

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$$\left(\sqrt{1/2}\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(s^2+t^2)}\, ds\, dt$$

So,

$r\cos\theta = s \qquad t = r\sin\theta$

$s \to 0 \Rightarrow r \to 0$

$s \to \infty \Rightarrow r \to \infty$

also $s^2 + t^2 = r^2$

$\tan\theta = \frac{t}{s} \Rightarrow \theta = \tan^{-1}\left(\frac{t}{s}\right)$

And $ds\, dt = r\, dr\, d\theta$

So, $$\left(\sqrt{\tfrac{1}{2}}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

Let $r^2 = p$

$2r\, dr = dp$

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty \frac{e^{-p}}{2}\, dp\, d\theta$$

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 = 2\int_0^{\pi/2}\left[-e^{-p}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2}[-0+1]\, d\theta$$

$$= 2[\theta]_0^{\pi/2} = 2\times\pi/2$$

$$\sqrt{\tfrac{1}{2}} = \sqrt{\pi}$$

Ex. 4 Evaluate

a) $\int_0^\infty x^5 e^{-x}\, dx$

$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$

so, $\int_0^\infty x^5 e^{-x}\, dx = \int_0^\infty e^{-x} x^{6-1}\, dx$ $\qquad \Gamma n = (n-1)!$

$= \Gamma 6$

$= 5!$

$= 120$

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b) $\int_0^\infty x^6 e^{-2x} dx$

$$\frac{\overline{|n}}{a^n} = \int_0^\infty e^{-at} t^{n-1} dt$$

So, $\int_0^\infty e^{-2x} x^6 dx = \int_0^\infty e^{-2x} x^{7-1} dx$ $\quad \because a=2$, $\because n=7$

$$= \frac{\overline{|7}}{2^7}$$

$\overline{|n} = (n-1)!$

$$= \frac{6!}{2^7}$$

$$= 45$$

c) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$

$x^3 = y$ $\quad x = y^{1/3}$

$3x^2 dx = dy$

$$\int_0^\infty y^{1/6} e^{-y} \frac{dy}{3y^{2/3}}$$

$$= \frac{1}{3}\int_0^\infty y^{-1/2} e^{-y} dy$$

$$= \frac{1}{3}\int_0^\infty y^{1/2-1} e^{-y} dy$$

$$= \frac{1}{3}\overline{|1/2}$$

$$= \frac{\sqrt{\pi}}{3}$$

Q $\int_0^\infty \sqrt[4]{x} = e^{-\sqrt{x}} dx$

let $\sqrt{x} = t$

$x = t^2$

$\frac{1}{2\sqrt{x}} dx = dt$

$dx = 2t\, dt$

as $x \to 0$ $\quad t \to 0$

$x \to \infty$ $\quad t \to \infty$

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Now, $\int_0^\infty t^{1/2} e^{-t} 2t \, dt$

$= 2\int_0^\infty e^{-t} t^{3/2} dt$

$= 2\int_0^\infty e^{-t} t^{5/2-1} dt$

$= 2 \Gamma 5/2$

$= 2 \times \frac{3}{2} \times \frac{1}{2} \Gamma 1/2$

$\Gamma n = (n-1)!$

$\Gamma n = (n-1)\Gamma n-1$

$= \frac{3}{2}\sqrt{\pi}$

Q $\int_0^1 x^{n-1} \left[\log\left(\frac{1}{x}\right)\right]^{m-1} dx$

Let $\log\left(\frac{1}{x}\right) = t$

$x \to 0 \Rightarrow t \to \infty$

$x \to 1 \Rightarrow t \to 0.$

$\frac{1}{x} = e^t$

$\infty = e^t$

$e^\infty = \infty$

$\frac{1}{x} = e^{+t}$

$x = e^{-t}$

$dx = -e^{-t} dt$

$I = \int_\infty^0 (e^{-t})^{n-1} t^{m-1} (-e^{-t}) dt$

$= \int_0^\infty e^{-nt} t^{m-1} dt$

$\because \int_a^b f(x) = -\int_b^a f(x)$

$= \int_0^\infty e^{-nt} t^{m-1} dt$

$= \frac{\Gamma m}{n^m}$

$\because \frac{\Gamma n}{a^n} = \int_0^\infty e^{-at} t^{n-1} dt$

OR

$I = \int_0^\infty e^{-nt} t^{m-1} dt$

$nt = u$

$n\,dt = du$

$I = \int_0^\infty e^{-u} \left(\frac{u}{n}\right)^{m-1} \frac{du}{n}$

$= \int_0^\infty \frac{1}{n^m} e^{-u} u^{m-1} du$

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$$= \frac{\Gamma m}{n^m} \qquad \Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{\Gamma m}{n^m}$$

Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad —— ①$$

$m>0 \quad n>0$

eq. ① is also known as Euler Beta function.

Properties of Beta function

① SYMMETRY

$$\beta(m,n) = \beta(n,m)$$

We know that

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1} (x)^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$= \beta(n,m)$$

② β function in form of Trigonometric functions.

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\because \beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x \to 0 \quad \theta \to 0$

$x \to 1 \quad \theta \to \pi/2$

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$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} (2\sin\theta\cos\theta)\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

H.P.

Let $2m-1 = p$

$m = \frac{p+1}{2}$

so, $\int_0^{\pi/2} \sin^m\theta \cos^n\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

(3) In terms of improper integrals.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Put $x = \frac{y}{1+y}$

$$dx = \frac{(1+y) - y}{(1+y)^2} dy = \frac{1}{(1+y)^2} dy$$

$$x = \frac{y}{1+y}$$

$$x(1+y) = y$$

$$\Rightarrow x + xy - y = 0$$

$$\Rightarrow x + (x-1)y = 0$$

$$x = (1-x)y$$

as $x \to 0 \Rightarrow y \to 0$

$x \to 1 \Rightarrow y \to \infty$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(1 - \frac{y}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}} dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$\because \beta(m,n)$ is symmetric

$$\beta(n,m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dx = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

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(4) Relation Between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

∵ We know that

$$\frac{\Gamma n}{a^n} = \int_0^\infty e^{-at} \, t^{n-1} \, dt$$

$$\Gamma n = \int_0^\infty a^n e^{-at} \, t^{n-1} \, dt$$

Multiply both sides by $e^{-a} a^{m-1}$

$$\Gamma n \, e^{-a} a^{m-1} = \int_0^\infty a^{m+n-1} e^{-a(t+1)} \, t^{n-1} \, dt$$

Integrating both sides b/w the limits 0 to ∞

$$\Gamma n \int_0^\infty e^{-a} a^{m-1} \, da = \int_0^\infty \int_0^\infty a^{m+n-1} e^{-a(t+1)} \, t^{n-1} \{dt\} \, da$$

$$= \int_0^\infty t^{n-1} \left\{ \int_0^\infty a^{m+n-1} e^{-a(t+1)} \, da \right\} dt$$

Let $a(1+t) = y$

as $t \to 0$ $\quad y \to 0$

$t \to \infty$ $\quad y \to \infty$

$(1+t) \, da = dy$

So,

$$\Gamma n \int_0^\infty e^{-a} a^{m-1} \, da = \int_0^\infty t^{n-1} \left\{ \int_0^\infty \left(\frac{y}{1+t}\right)^{m+n-1} e^{-y} \frac{dy}{(1+t)} \right\} dt$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} \left\{ \int_0^\infty y^{m+n-1} e^{-y} \, dy \right\} dt$$

$$∵ \Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\Gamma n \, \Gamma m = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} \left\{ \Gamma m+n \right\} dt$$

$$= \Gamma m+n \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} \, dt$$

$$\Gamma n \, \Gamma m = \Gamma m+n \; \beta(n,m)$$

$$∵ \beta(m,n) = \beta(n,m)$$

~~$\Gamma n \, \Gamma m = \Gamma m+n \; \beta$~~

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Gamma Formula

$$2\int_0^{\pi/2} \sin^m\theta \cos^m\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{\Gamma\frac{m+n+2}{2}}$$

$$\therefore 2\int_0^{\pi/2} \sin^m\theta \cos^m\theta = \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Euler's Functional Equation Euler's Functional Equation

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad 0<n<1$$

n is not integer

$n = 1/2$

$$\Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{\pi}{1}$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

Important Relation in Beta Function

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

Proof: $\because$ we know that

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

RHS :- $\beta(m+1,n) + \beta(m,n+1)$

$$= \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)}$$

Since $\Gamma(n+1) = n\Gamma n$

$$= \frac{m\Gamma m\,\Gamma n + n\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{(m+n)\,\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)}$$

$$= \beta(m,n)$$

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Legendre's Duplication Formula

Duplication Formula for Gamma Functions.

Statement: Let $m \in N$ then

$$\Gamma m \, \Gamma\frac{m+1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$$

Proof: ∵ We know that

$$\frac{\beta(m,n)}{2} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$= \frac{\Gamma m \Gamma n}{2\Gamma m+n} \quad \text{——— (1)}$$

Put $2n-1=0$ in (1), we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma 1/2}{2\Gamma m+1/2} \quad \text{——— (2)}$$

Again put $n=m$ in (1), we have

$$\int_0^{\pi/2} (\sin\theta \cos\theta)^{2m-1} = \frac{(\Gamma m)^2}{2\Gamma 2m} \quad \text{——— (3)}$$

Now multiply and divide eq. (3) by 2^{2m-1}

$$\frac{1}{2^{2m-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

Put $2\theta = \psi$ as $\theta \to 0 \Rightarrow \psi \to 0$

$d\theta = d\psi/2$ $\theta \to \pi/2 \Rightarrow \psi \to \pi$

$$\Rightarrow \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin\psi)^{2m-1} \frac{d\psi}{2} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

if $f(a-x) = f(x)$

then $\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$

||| $\sin(\pi-\psi) = \sin\psi$

$$\int_0^{\pi} \sin^{2m-1}\psi \, d\psi = 2\int_0^{\pi/2} \sin^{2m-1}\psi \, d\psi$$

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$$\frac{1}{2^{2m-1}} \cdot 2 \cdot \frac{1}{2}\int_0^{\pi/2} \sin^{2m-1}\psi \, d\psi = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

From (ii),

$$\frac{1}{2^{2m-1}} \quad 2 \cdot \frac{1}{2}\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

From (2),

$$\frac{1}{2^{2m-1}} \quad \frac{m\pi}{2\Gamma m+\frac{1}{2}} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$$

H.P.

Note

I $\int_0^\infty e^{-ax} \sin bx \, x^{n-1} dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \sin n\theta$

II $\int_0^\infty e^{-ax} \cos bx \, x^{n-1} dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \cos n\theta$

where $\theta = \tan^{-1}(b/a)$

We know $\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma n}{a^n}$

Replace a by z

$$\int_0^\infty x^{n-1} e^{-zx} dx = \frac{\Gamma n}{z^n}$$

Take $z = a - ib$

$$\int_0^\infty e^{-(a-ib)x} x^{n-1} dx = \frac{\Gamma n}{z^n} = \frac{\Gamma n}{(a-ib)^n} = \frac{\Gamma n \, (a+ib)^n}{(a^2+b^2)^n}$$

let $a = r\cos\theta$

$b = r\sin\theta$

$r = \sqrt{a^2+b^2}$ $\qquad \theta = \tan^{-1}(b/a)$

$$\int_0^\infty e^{-ax} e^{ibx} x^{n-1} dx = \Gamma n \frac{\cancel{(a^2+b^2)^{n/2}} \cancel{(\cos\theta + i\sin\theta)^n} (r\cos\theta + ir\sin\theta)^n}{(a^2+b^2)^n}$$

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$$\int_0^\infty e^{-(a-ibx)} x^{n-1}\,dx = \frac{\Gamma n}{(a^2+b^2)^n}(r\cos\theta + ir\sin\theta)^n$$

$$= \frac{\Gamma n}{(a^2+b^2)^n} r^n (\cos\theta + i\sin\theta)^n$$

$$\int_0^\infty e^{-ax} e^{+ibx} x^{n-1}\,dx = \frac{\Gamma n\,(a^2+b^2)^{n/2}}{(a^2+b^2)^n}(\cos\theta + i\sin\theta)^n$$

$$\int_0^\infty e^{-ax}(\cos bx + i\sin bx)\,x^{n-1}\,dx = \frac{\Gamma n\,(\cos\theta + i\sin\theta)^n}{(a^2+b^2)^{n/2}}$$

$$\Rightarrow \int_0^\infty e^{-ax}(\cos bx + i\sin bx)\,x^{n-1}\,dx = \frac{\Gamma n\,[\cos n\theta + i\sin n\theta]}{(a^2+b^2)^{n/2}}$$

Compare real and imaginary part.

$$\int_0^\infty e^{-ax}\sin bx\; x^{n-1}\,dx = \frac{\Gamma n \sin n\theta}{(a^2+b^2)^{n/2}}$$

$$\int_0^\infty e^{-ax}\cos bx\; x^{n-1}\,dx = \frac{\Gamma n \cos n\theta}{(a^2+b^2)^{n/2}}$$

Case: Put $a = 0$ so that $\theta = \pi/2$

$$\int_0^\infty x^{n-1}\sin bx = \frac{\Gamma n}{b^n}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_0^\infty x^{n-1}\cos bx = \frac{\Gamma n}{b^n}\cos\left(\frac{n\pi}{2}\right)$$

Imp. Result

$$\int_0^\infty x^{n-1}\sin bx = \frac{\Gamma n}{b^n}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_0^\infty x^{n-1}\cos bx = \frac{\Gamma n}{b^n}\cos\left(\frac{n\pi}{2}\right)$$

$$\int_0^\infty \cos bx^{1/n}\,dx = \frac{\Gamma n+1}{b^n}\cos\frac{n\pi}{2}$$

Formula On The Successive Product of Gamma function

For any $n \in N$ and $n > 1$

$$\Gamma\frac{1}{n}\,\Gamma\frac{2}{n}\,\Gamma\frac{3}{n}\cdots\Gamma\frac{n-1}{n} = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

Thm : For any n element of N ($n \in N$) and $n > 1$

$$\Gamma\frac{1}{n}\,\Gamma\frac{2}{n}\,\Gamma\frac{3}{n}\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

Evaluate the following

a) $\Gamma\left(\frac{-1}{2}\right)$ b) $\Gamma\frac{-5}{2}$ c) $\Gamma\left(\frac{-7}{2}\right)$

$\because$ We

$$\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$$

$$\Gamma\frac{-1}{2}\,\Gamma\left(1-\left(\frac{-1}{2}\right)\right) = \frac{\pi}{\sin\frac{-\pi}{2}}$$

$$\Gamma\frac{-1}{2}\,\Gamma\frac{3}{2} = \frac{\pi}{\sin\left(\frac{-\pi}{2}\right)}$$

$$\Gamma n = (n-1)\,\Gamma n-1$$

$$\Gamma 3/2 = \left(\frac{3}{2}-1\right)\Gamma\frac{1}{2} \qquad = \sqrt{\pi}$$

$$= \frac{1}{2}\sqrt{\pi}$$

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$$\Gamma\left(-\frac{1}{2}\right) \frac{\sqrt{\pi}}{2} = \frac{\pi}{-\sin\left(\frac{\pi}{2}\right)}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{2\sqrt{\pi}}{-1}$$

$$= -2\sqrt{\pi}$$

1) $$\Gamma\left(-\frac{5}{2}\right)\Gamma\left(1+\frac{5}{2}\right) = \frac{\pi}{-\sin\frac{5\pi}{2}}$$

$$\Gamma\left(-\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right) = \frac{\pi}{-\sin\left(2\pi+\frac{\pi}{2}\right)}$$

$$\Gamma\left(-\frac{5}{2}\right)\left(\frac{5}{2}\times\frac{3}{2}\sqrt{\pi}\right) = \frac{\pi}{-\sin\pi/2}$$

$$= -\frac{4}{15}\sqrt{\pi}$$

1) $$\Gamma\left(-\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right) = \frac{\pi}{-\sin\left(\frac{7\pi}{2}\right)}$$

$$= \frac{\pi}{-\sin\left(3\pi+\frac{\pi}{2}\right)}$$

$$= \pi$$

Q T.P.

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$$

$\because$ We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \text{—①}$$

Now LHS :- $\int_0^\infty x^8 \frac{(1-x^6)}{(1+x)^{24}}\,dx$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9) \quad (\text{from ①})$$

$= 0$

$=$ RHS. since $\beta(m,n) = \beta(n,m)$

Q Show that

$$\int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2} \sec\left(\frac{n\pi}{2}\right)$$

$\because$ We know that

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \quad \text{—①}$$

and $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ —②

Now LHS: $\int_0^{\pi/2} \tan^n x\,dx$

$$= \int_0^{\pi/2} \sin^n x \cos^{-n} x\,dx$$

$$= \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{-n+1}{2}\right) \quad \text{By ①}$$

$$\frac{1}{2} \cdot \frac{\Gamma\frac{n+1}{2}\,\Gamma\left(1-\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+1-\frac{n+1}{2}\right)} \quad \text{by } (2)$$

$$= \frac{1}{2}\,\Gamma\frac{n+1}{2}\,\Gamma\frac{1-n}{2} \qquad \because\ 1-\frac{n+1}{2} = \frac{2-n-1}{2} = \frac{1-n}{2}$$

$$= \frac{1}{2}\,\Gamma\frac{n+1}{2}\,\Gamma\left(1-\left(\frac{n+1}{2}\right)\right)$$

$$\because\ \&\ \Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\pi}{\cos\left(\frac{n\pi}{2}\right)} \qquad \because\ \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$= \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$$

$= RHS.$

Q Tip.

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma 1/n}{n\,\Gamma\frac{1}{n}+\frac{1}{2}}$$

$\because$ We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$1-x^n = y \quad \Rightarrow \quad x = (1-y)^{1/n}$$

$$-n(x^{n-1})\,dx = dy$$

as $x \to 0$ $y \to 1$

as $x \to 1$ $y \to 0$

$$\Rightarrow dx = \frac{-1}{n}\,\frac{dy}{\left((1-y)^{1/n}\right)^{n-1}}$$

$\Rightarrow \quad dx = -\frac{1}{n}(1-y)^{\frac{1}{n}-1}\, dy$

LHS. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= -\int_1^0 \frac{1}{n}(1-y)^{1/n-1} \frac{1}{\sqrt{y}}\, dy$

$= +\frac{1}{n}\int_0^1 (1-y)^{1/n-1}\, y^{-1/2}\, dy$

$= \frac{1}{n}\int_0^1 y^{1/2-1}(1-y)^{1/n-1}\, dy$

$= \frac{1}{n}\beta\left(\frac{1}{2}, \frac{1}{n}\right)$

$= \frac{1}{n}\frac{\Gamma\frac{1}{2}\,\Gamma 1/n}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$

$= \frac{1}{n}\frac{\sqrt{\pi}\,\Gamma(1/n)}{\Gamma\left(1/2+\frac{1}{n}\right)}$

$=$ RHS.

Ex:- Prove that $I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a\cos^4\theta + b\sin^4\theta}} = \frac{\left(\Gamma 1/4\right)^2}{4(ab)^{1/4}\sqrt{\pi}}$

Put $\tan\theta = t$

$\sec^2\theta\, d\theta = dt$

as $\theta \to 0 \Rightarrow t \to 0$

as $\theta \to \pi/2 \Rightarrow t \to \infty$

LHS:- $\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sqrt{a + b\tan^4\theta}}$

$= \int_0^{\infty} \frac{dt}{\sqrt{a+bt^4}}$

Again put $bt^4 = ay \Rightarrow 4bt^3\, dt = a\, dy$

$\Rightarrow dt = \frac{a}{4b\cdot\left(\frac{ay}{b}\right)^{3/4}}\, dy$

as $t \to 0 \Rightarrow y = 0$

as $t \to \infty \Rightarrow y = \infty$

LHS:- $\int_0^\infty \frac{a}{4b \left(\frac{ay}{b}\right)^{3/4} \sqrt{a}\ (1+y)^{1/2}} dy$

$= \frac{a^{1/2}\, a^{-3/4}}{4\, b^{\frac{(4-3)}{4}}} \int \frac{dy}{y^{3/4}\ (1+y)^{1/2}}$

$= \frac{a^{-1/4}\, b^{-1/4}}{4} \int_0^\infty \frac{y^{-3/4}}{(1+y)^{1/2}} dy$

$= \frac{1}{4(ab)^{1/4}} \int_0^\infty \frac{y^{1/4-1}}{(1+y)^{1/4+1/4}} dy$

$= \frac{1}{4\ (ab)^{1/4}} \beta\left(\frac{1}{4}, \frac{1}{4}\right).$

$\because \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

Also. $\beta(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma m+n}$

$= \frac{1}{4\ (ab)^{1/4}} \frac{\Gamma 1/4\ \Gamma 1/4}{\Gamma 1/4+1/4}$

$= \frac{1}{4\ (ab)^{1/4}} \frac{(\Gamma 1/4)^2}{\Gamma 1/2}$

$= \frac{1}{4\ (ab)^{1/4}} \frac{(\Gamma 1/4)^2}{\sqrt{\pi}}$

= RHS.

Ex → T.P. $\int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\pi/2}$

$\because$ We know $\int_0^\infty x^{m-1} \sin bx\, dx = \frac{\Gamma m}{b^m} \sin\left(\frac{m\pi}{2}\right)$

LHS (1) $\int_0^\infty \sin(x^2)\, dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{dt}{2x}$

$$dx = \frac{dt}{2\sqrt{t}}$$

as $x \to 0 \Rightarrow t \to 0$

as $x \to \infty \Rightarrow t \to \infty$

LHS $= \int_0^\infty \frac{1}{2\sqrt{t}} \sin t \, dt$

$= \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} \sin t \, dt$

Since $\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma m}{b^m} \sin \frac{m\pi}{2}$

$m = 1/2 \qquad b = 1$

LHS $= \frac{1}{2} \times \frac{\Gamma 1/2}{1^{1/2}} \sin \frac{\pi}{4}$

$= \frac{\sqrt{\pi}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{2}\sqrt{\frac{\pi}{2}}$

$=$ RHS.

Ex: T.P. $\beta(p,q) = a^p b^q \int_0^\infty \frac{x^{p-1}}{(ax+b)^{p+q}} dx$

Hence deduce that

$$\int_0^{\pi/2} \frac{\sin^{2p-1}\theta \cos^{2q-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{p+q}} d\theta = \frac{\beta(p,q)}{2a^p b^q}$$

RHS: $ax = by \Rightarrow x = \frac{by}{a}$

as $x \to 0 \Rightarrow y \to 0$ and $x \to \infty \Rightarrow y \to \infty$

RHS $= a^p b^q \int_0^\infty \left(\frac{by}{a}\right)^{p-1} \frac{1}{(by+b)^{p+q}} \frac{b}{a} dy$

$= a^p b^q \frac{b}{a} \int_0^\infty \left(\frac{b}{a}\right)^{p-1} \frac{y^{p-1}}{b^{p+q}(1+y)^{p+q}} dy$

$$= a^{p-1-p+1}\, b^{q+1+p-1-p-q} \int_0^\infty \frac{y^{p-1}}{b^{p+q}\,(1+y)^{p+q}}\, dy$$

$$= \int_0^\infty \frac{y^{p-1}}{b^{p+q}\,(1+y)^{p+q}}\, dy = \beta(p,q) = LHS.$$

H.P.

$$\cancel{\beta(p,q) = a^p b^q \int_0^{\pi/2} \frac{(\tan^2\theta)^{p-1}}{(a\tan^2\theta + b)^{p+q}}\, 2\tan\theta \sec^2\theta\, d\theta}$$

$$\cancel{\frac{\beta(p,q)}{2a^p b^q}}$$

$$\beta(p,q) = a^p b^q \int_0^\infty \frac{x^{p-1}}{(ax+b)^{p+q}}\, dx \quad \text{——— (1)}$$

Put $x = \tan^2\theta \Rightarrow dx = 2\tan\theta \sec^2\theta\, d\theta$

When $x=0$ $\theta=0$ $x=\infty$, $\theta = \pi/2$

$$\beta(p,q) = a^p b^q \int_0^{\pi/2} \frac{(\tan^2\theta)^{p-1}}{(a\tan^2\theta + b)^{p+q}}\, 2\tan\theta \sec^2\theta\, d\theta$$

$$= b^q a^p \int_2^{\pi/} \frac{\sin^{2p-2} \cos^{-2p+2} \sin\theta\, (\cos^2\theta)^{p+q}}{(a\sin^2\theta + b\cos^2\theta)^{p+q} \cos^2\theta \cos\theta}\, d\theta$$

$$\beta(p,q) = a^p b^q \int_0^{\pi/2} \frac{\sin^{2p-1}\theta \cos^{2q-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{p+q}}\, d\theta$$

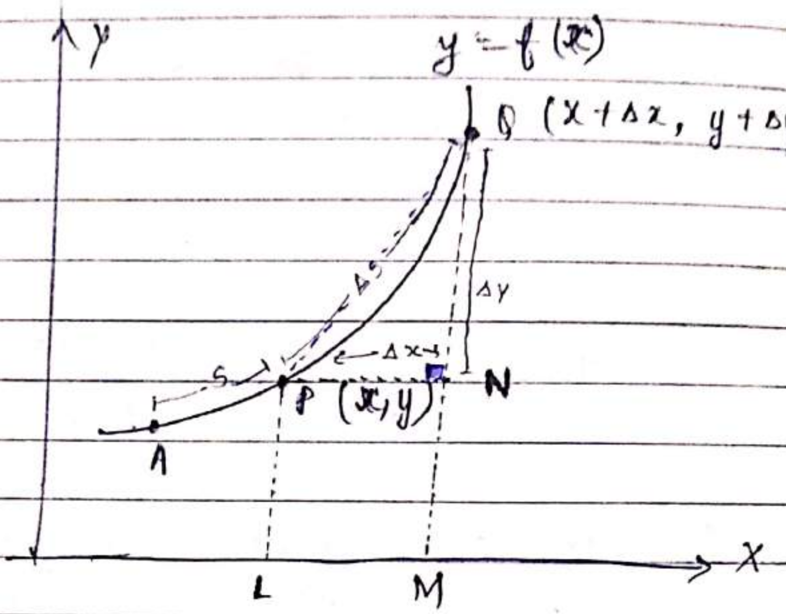
$$= \cancel{\frac{\beta(p,q)}{2a^p b^q}}$$

$$\frac{\beta(p,q)}{a^p b^q} = \int_0^{\pi/2} \frac{\sin^{2p-1} \cos^{2q-1}}{(a\sin^2\theta + b\cos^2\theta)^{p+q}}\, d\theta$$

H.P.

Rectification

The method of finding the length b/w 2 points on any plane curve is called Rectification.



Lengths of Curves :- Let $y = f(x)$ be any curve and A as a fixed point on the curve. Let 2 nearest point on curve $P(x, y)$ & $Q(x+\Delta x, y+\Delta y)$ which has curve length from s and $s+\Delta s$ respectively.

∴ Arc length $PQ = \Delta s$

Now draw perpendicular from P & Q as PL & QM respectively. Draw a perpendicular PN on QM.

∴ acc. to figure

$$PN = \Delta x$$

$$QN = \Delta y$$

PQ is the chord of the curve.

Now, in ΔPQN

$$PQ^2 = (PN)^2 + (QN)^2$$

$$\Rightarrow (PQ)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \left(\frac{PQ}{arc\ PQ} \cdot \frac{arc\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \left(\frac{PQ}{arc\ PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

As $Q \to P$, Δx , $\Delta y \to 0$

$\therefore$

$$\lim_{Q \to P} \frac{PQ}{\text{arc } PQ} = 1$$

Taking limit on both sides
as $Q \to P$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \qquad \because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{ds}{dx} = \pm\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Let s increases when x increases

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

On integrating from lim a to b

$$\int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$b - a = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\boxed{\text{arc } PQ = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Expression for Arc Length

1. Cartesian Form:- If the curve eq. is $y = f(x)$ then curve length b/w 2 points which lie on curve i.e. $a \le x \le b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (a)$$

Corr. If curve eq. is $x = f(y)$ and $c \le y \le d$ then

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{--- (b)}$$

2. Parametric Form :- If curve eq. is $x = x(t)$ and $y = y(t)$ such that $t_1 \le t \le t_2$ then

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{--- (c)}$$

$$s = \int_{t_1}^{t_2} \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}}\, dx = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{dx/dt}\, dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3. Polar form If the curve eq. be $r = f(\theta)$ and $\theta_1 \le \theta \le \theta_2$ then

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{--- (d)}$$

Corr. If the curve eq. in polar form be $\theta = f(r)$ and $r_1 \le r \le r_2$ then

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr \quad \text{--- (e)}$$

Note To memorise above formulae

1. $(ds)^2 = (dx)^2 + (dy)^2$ — for cartesian divide by $(dx)^2$ for a; divide by $(dy)^2$ for b

2. $(ds)^2 = (dr)^2 + (r\,d\theta)^2$ — divide by $d\theta^2$ for (d); divide by $(dr)^2$ for (e)

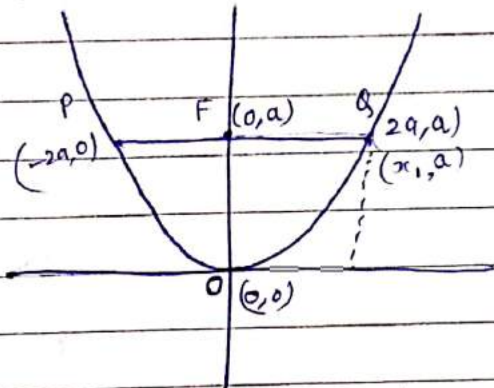
in form of p, r

4. Pedal Equation Form $p = f(r)$

If the curve eq. $p = f(r)$ and two points A and B on curve are radius vector r_1 and r_2 respectively, then

$$S = \int_{r_1}^{r_2} \frac{r\,dr}{\sqrt{r^2 - p^2}}$$

Ex- Prove that the length of the arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of the latus rectum is $a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$



$$S = \int_0^{2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$x^2 = 4ay$

$2x = 4a \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{2a}$

$x^2 = 4ay$
Put $y = a$
$x^2 = 4a^2$
$\cancel{x = \pm 2a}$
So, $B(2a, 0)$
$P(-2a, 0)$

$$S = \int_0^{2a} \sqrt{1 + \left(\frac{x}{2a}\right)^2}\, dx$$

Since $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c$

$$S = \frac{1}{2a}\int_0^{2a} \sqrt{(2a)^2 + x^2}\, dx$$

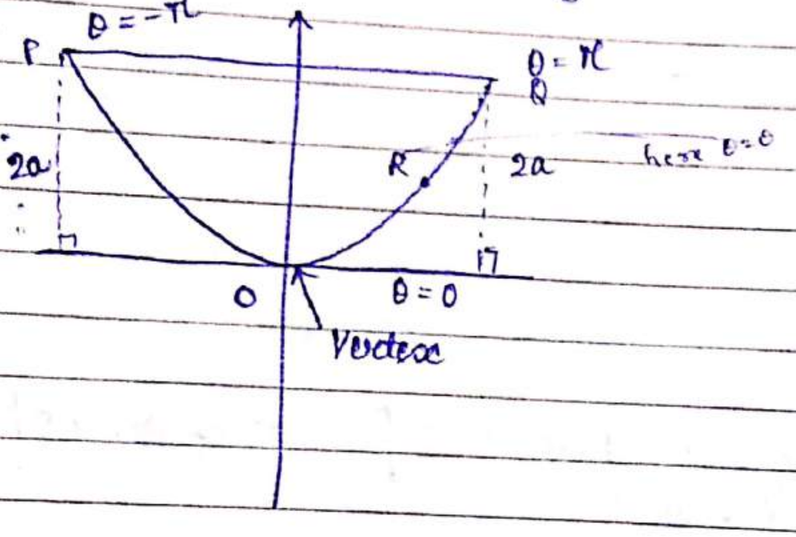
$$= \frac{1}{2a}\left[\frac{x}{2}\sqrt{x^2+4a^2} + \frac{4a^2}{2}\log x + \sqrt{x^2+4a^2}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[a(2a)\sqrt{2} + 2a^2\log(2a + 2a\sqrt{2}) - 2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[2a^2\sqrt{2} + 2a^2\log(1+\sqrt{2})\right] = a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Ex Show that the length of the arc from the vertex to any pt. on the cycloid $x = a(\theta + \sin\theta)$ — ① $y = a(1 - \cos\theta)$ — ② is $\sqrt{8ay}$. Also show that whole length of an arc of the curve is $8a$.

By curve tracing, the curve of the cycloid is



In given fig. O(0,0) is vertex and let R be any point on the cycloid.

and given curve

$$x = a(\theta + \sin\theta) \text{ — ① } \Rightarrow dx/dt = a(1 + \cos\theta)$$
$$y = a(1 - \cos\theta) \text{ — ② } \Rightarrow dy/dt = a\sin\theta$$

∴ length of arc OR is

$$s = \int_{\theta=0}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \int_0^{\theta} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \int_0^{\theta} a \sqrt{2 + 2\cos\theta} = \sqrt{2}\, a \int_0^{\theta} \sqrt{(1+\cos\theta)}\, d\theta$$

$$= 2\sqrt{2}a(\ldots - \sin\ldots)$$

$$= \sqrt{2}a \int_0^{\theta} \sqrt{2} \cos\frac{\theta}{2} \, d\theta$$

$$s = 2a \left[2 \sin\left(\frac{\theta}{2}\right) \right]_0^{\theta} \quad \text{——} (3)$$

$$= 4a \sin\frac{\theta}{2}$$

$$y = a(1-\cos\theta)$$
$$y = a \, 2\sin^2\frac{\theta}{2}$$
$$\sqrt{\frac{y}{2a}} = \sin\frac{\theta}{2}$$

$$= 4a\sqrt{\frac{y}{2a}}$$

$$= \sqrt{\frac{16a^2}{2a} y}$$

$$s = \sqrt{8ay}$$

From (2)

Now length of whole curve POQ = 2 [length of Arc OQ]

$$= 2 \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 2\left\{4a \sin\left(\frac{\theta}{2}\right)\right\}_0^{\pi} \quad \text{From eq. (3)}$$

$$= 8a \left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= 8a$$

Ex 7 Prove that whole length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $6a$.

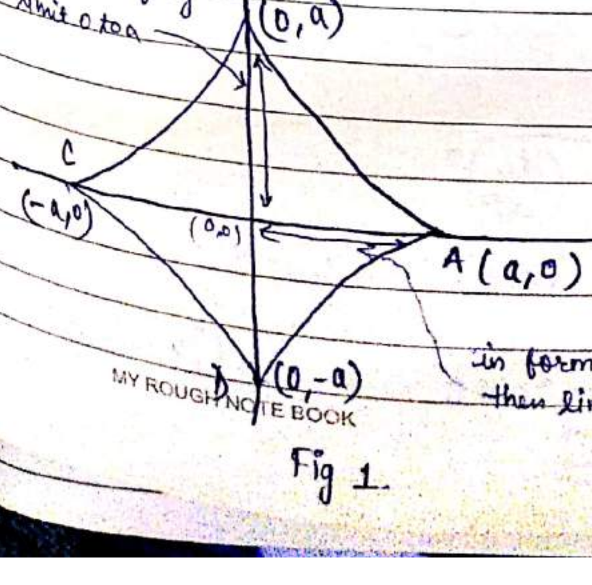


Fig 1.

Given eq.

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{——— (1)}$$

After curve tracing, the resultant fig. is Fig 1

Now, differentiating w.r.t x

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

$$\frac{-x^{-1/3}}{y^{-1/3}} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Therefore whole curve length = 4 [length of curve AB]

$$= 4\left[\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx\right]$$

$$= 4\left[\int_0^a \sqrt{1+\left(\frac{y}{x}\right)^{2/3}}\, dx\right]$$

$$= 4 \int_0^a \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}\, dx$$

$$= 4 \int_0^a \frac{\sqrt{a^{2/3}}}{x^{1/3}}\, dx$$

$$= 4 \int_0^a a^{1/3} x^{-1/3}\, dx$$

$$= 4 \times \left[\frac{3x^{2/3}}{2} \quad a^{1/3}\right]_0^a$$

$$= 6 \left[x^{2/3}\right]_0^a a^{1/3}$$

$$= 6\, a^{2/3} a^{1/3}$$

$$= 6a$$

HW

Q Ex: Find the perimeter of the cardioid $r = a(1+\cos\theta)$ & prove that the upper half arc of the cardioid is bisected by the line $\theta = \pi/3$

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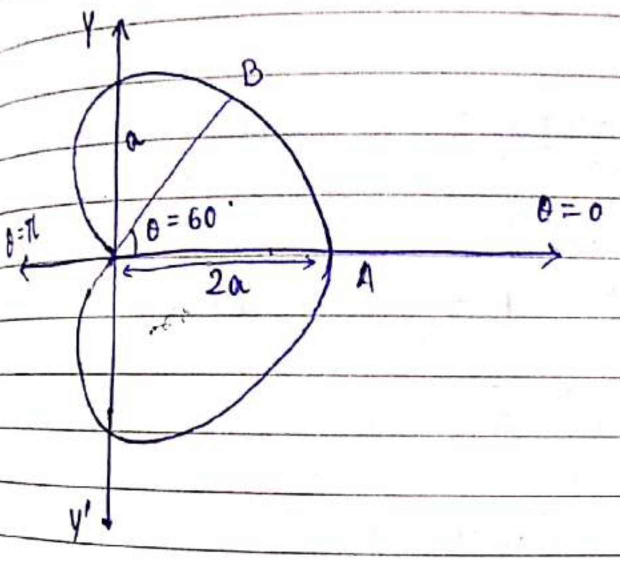
Q.2 Prove that the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from its vertex to the pt (a,a) is $\frac{a}{27}\left[13\sqrt{13} - 8\right]$

$\frac{\pi}{6} = \frac{1}{2}$

us

Home work. Solutions

Q.1 Find the perimeter of the cardioid $r = a(1+\cos\theta)$ ....



By curve tracing, ~~tra~~ trace the curve.

Curve is symmetrical about Initial line.

Differentiating wrt θ

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$\therefore$ length of upper half.

$$= \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{(a(1+\cos\theta))^2 + (-a\sin\theta)^2}\, d\theta$$

$$= a\int_0^\pi \sqrt{2(1+\cos\theta)}\, d\theta = 2a\int_0^\pi \cos\frac{\theta}{2}\, d\theta$$

$$= 2a\left[2\sin\frac{\theta}{2}\right]_0^\pi = 4a \quad \text{--- (1)}$$

So, req. perimeter $= 2 \times 4a = 8a$ --- (2)

For finding length of arc from A $(\theta = 0)$ to B $(\theta = 60^\circ)$ in upper half, replace limits π by $\pi/3$

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$\sin\frac{\pi}{6} = \frac{1}{2}$

$$AB = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3} = 4a\sin\frac{\pi}{6}$$

$$AB = 2a$$

which is half of that of upper half. H.P.

Q.2 Differentiating eq. of curve

$$2ay\cdot\frac{dy}{dx} = 3x^2$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{9}{4a^2}\cdot\frac{x^4}{y^2} = \frac{9x^4}{4a^2}\cdot\frac{a}{x^3} = \frac{9x}{4a} \quad \text{— (1)}$$

Req. length

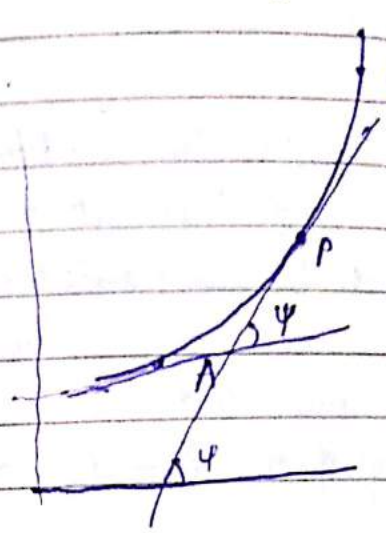
$$= \int_0^a \sqrt{\left(1+\frac{9x}{4a}\right)}\,dx = \frac{1}{2\sqrt{a}}\int_0^a \sqrt{4a+9x}\,dx$$

$$= \frac{1}{2\sqrt{a}}\left[\frac{(4a+9x)^{3/2}}{9}\cdot\frac{2}{3}\right]_0^a$$

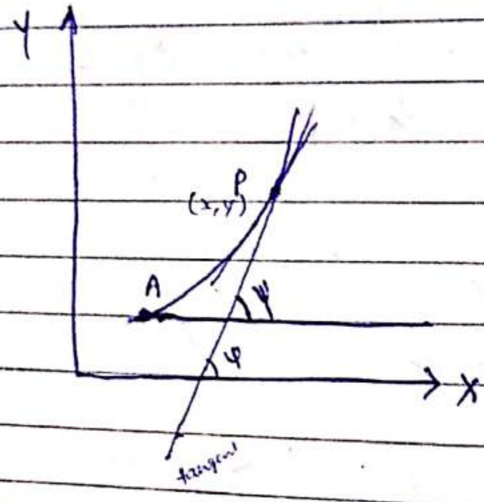
$$= \frac{1}{27}\left\{13\sqrt{13}-8\right\}a$$

Intrinsic Equation of a Curve

Intrinsic Relation between s and ψ is called Intrinsic eq. where s is the arc length from fixed point A to variable point P, and ψ is the angle b/w tangent on point A and P.



Intrinsic Eq. from the Cartesian Equation



P = variable pt

A = fixed point

If abscissa of A and P point are a and x respectively.

Let curve eq. $y = f(x)$ ——— ①

$\because$ we know that $\tan \psi = \frac{dy}{dx} = f'(x)$ ——— ②

Also we know that

$$s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^x \sqrt{1 + [f'(x)]^2}\, dx \quad \text{———} ③$$

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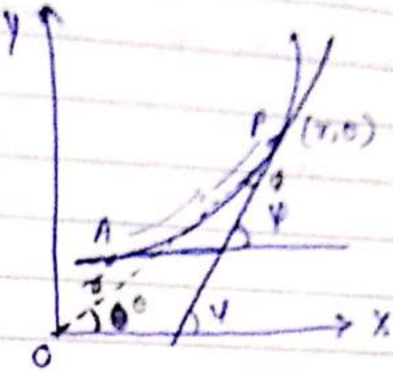
Eliminate x from (2) and (3) then the obtained relation is intrinsic eq of the given curve.

Intrinsic Equation From the Polar Equation



ϕ = angle b/w radial vector and tangent at P.

Let α be the ~~radial~~ vector angle at A.

∵ We know that

$$\psi = \theta + \phi \quad \text{——— (1)}$$

Let a curve eq. is $r = f(\theta)$ ——— (2)

Again since we know that $s = \int_{\alpha}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ — (3)

Also we know that

$$\tan\phi = r\frac{d\theta}{dr} \quad \text{——— (4)}$$

$$\Rightarrow \frac{r}{dr/d\theta} = \tan\phi$$

$$\Rightarrow \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)} \quad \text{——— (4)}$$

Now eliminate θ, ϕ from (1), (3) and (4) we obtain required intrinsic eq.

Parametric form [Intrinsic Eq. from the Parametric Eq.]

Let curve eq. in parametric form $x = f(t)$, $y = g(t)$ — (1)

∵ We know that $\tan\psi = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

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$$\Rightarrow \tan\psi = \frac{g'(t)}{f'(t)} \quad \text{——— (2)}$$

Also we know that

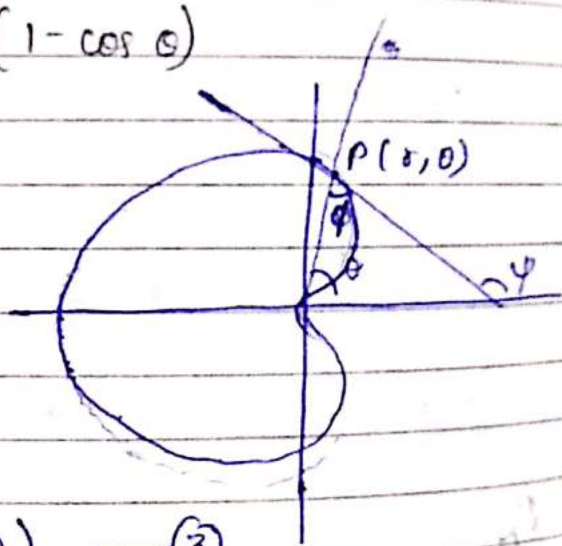
$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$s = \int_{t_1}^{t} \sqrt{f'(t) + g'(t)}\, dt \quad \text{——} \ (3)$$

Now eliminate t from (2) and (3) then the resultant eq is intrinsic equation.

Example. Find the intrinsic eq. of the cardioid ~~at~~

$$r = a(1-\cos\theta)$$



Given curve eq. $r = a(1-\cos\theta)$ —— (1)

∵ we know that

$$\psi = \theta + \phi \quad \text{——} \ (2)$$

From (1) $\frac{dr}{d\theta} = a\sin\theta$

and $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta}$ —— (3)

Also we know that

$$s = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta}\, d\theta$$

$$= a\sqrt{2} \int_0^{\theta} \sqrt{1-\cos\theta}\, d\theta$$

$$= a\sqrt{2}\sqrt{2} \int_0^{\theta} \sin\frac{\theta}{2}$$

$$= 4a \left[-\cos\frac{\theta}{2}\right]_0^{\theta}$$

$$s = 4a\left[1 - \cos\frac{\theta}{2}\right] = -4a\left[\cos\frac{\theta}{2} - 1\right]$$

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$$= 4a \cdot 2\sin^2\left(\frac{\theta}{4}\right)$$

$$S = 8a\sin^2\frac{\theta}{4} \quad \text{——— (4)}$$

From eq. (2), (3), (4) eliminate θ, ϕ

From (3) $\tan\phi = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{2\sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}$

$$\tan\phi = \tan\frac{\theta}{2}$$

$$\phi = \frac{\theta}{2}$$

from (2) $\psi = \theta + \phi$

$$= \theta + \frac{\theta}{2}$$

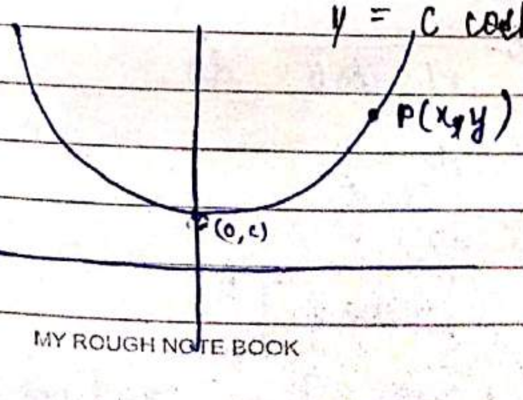
$$\Rightarrow \psi = \frac{3\theta}{2} \quad \text{——— (5)}$$

From (4), (5) $\Rightarrow \theta = \dfrac{2\psi}{3}$

$$S = 8a\sin^2\left(\frac{\psi}{6}\right) = 4a\left(1-\cos\frac{\psi}{3}\right)$$

which is intrinsic eq of the given curve.

Q. T.P the intrinsic eq. of the ~~curve~~. caternary $y = c\cosh\left(\frac{x}{c}\right)$ is $s = c\tan\psi$



$$y = \cosh\left(\frac{x}{c}\right)$$

$$\Rightarrow \tan\psi = \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right) \quad \text{--- (1)}$$

$$s = \int_0^x \sqrt{1 + \sinh^2\left(\frac{x}{c}\right)}\, dx = \int_0^x \sqrt{\cancel{1} + \cosh^2\left(\frac{x}{c}\right) \cancel{-1}}\, dx$$

$$= \int_0^x \cosh\left(\frac{x}{c}\right) dx = c \sinh\left(\frac{x}{c}\right)$$

from (1) $\quad s = c \tan\psi$

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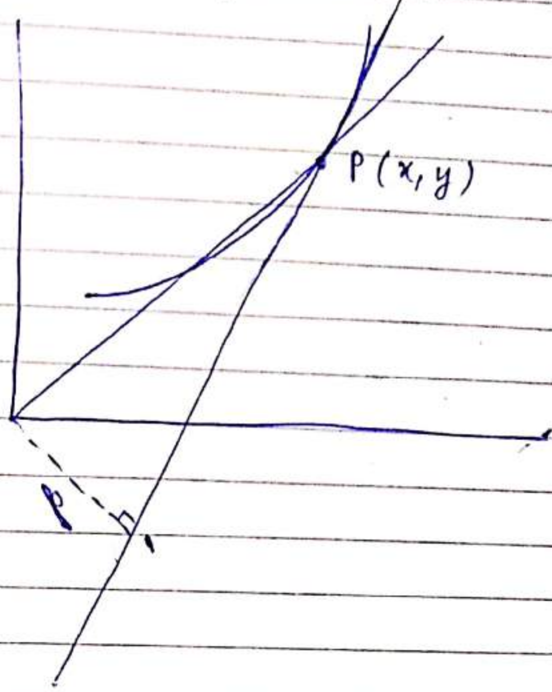
Q. Prove that the intrinsic eq. of the curve $p = r \sin\alpha$ is $s = a e^{\psi \cot\alpha}$ where a is arbitrary constant.

Given curve eq.

$$p = r \sin\alpha \quad \text{--- (1)}$$

$$\frac{dp}{dr} = \sin\alpha$$



$\because$ We know that arc length

$$s = \int_0^r \frac{r\,dr}{\sqrt{r^2 - p^2}}$$

$$= \int_0^r \frac{r\,dr}{r\sqrt{1-\sin^2\alpha}}$$

$$= \int_0^r \sec\alpha\,dr$$

$$= \sec\alpha\,(r - 0)$$

$$\Rightarrow s = r\sec\alpha \quad \text{--- (2)}$$

Again, $\because$ We know that

$$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp} = \frac{r}{\sin\alpha} \quad \text{--- (3)}$$

$$\frac{dr}{dp} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$$

so $$\frac{ds}{d\psi} = r\frac{dr}{dp} = r\operatorname{cosec}\alpha \quad \text{--- (4)}$$

By (2) & (4)

$$\frac{ds}{d\psi} = \frac{s}{\sec\alpha}\operatorname{cosec}\alpha = s\cot\alpha$$

$$\frac{ds}{s} = \cot\alpha\,d\psi$$

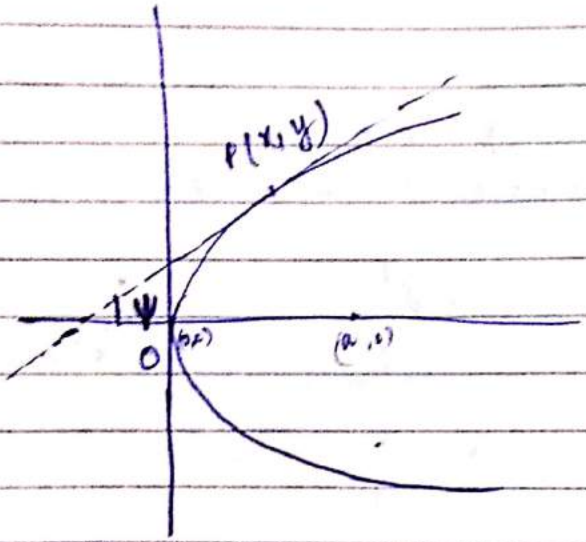
$$\Rightarrow \log s = \psi\cot\alpha + \log a$$

$$\Rightarrow \frac{s}{a} = e^{\psi\cot\alpha}$$

$$\Rightarrow s = a e^{\psi \cot\alpha} \qquad \text{H.P.}$$

which is intrinsic eq. ~~of the~~ given curve.

Q. Show that the intrinsic eq. of the parabola $y^2 = 4ax$ is
$s = a\cot\psi\,\mathrm{cosec}\,\psi + a\log(\cot\psi + \mathrm{cosec}\,\psi)$
ψ being the angle b/w the x axis and the tangent at the pt. whose distance from the vertex is s.



Given curve

$$y^2 = 4ax \quad \text{--- (1)}$$

$$2y = 4a\frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{y}{2a} = \cot\psi \quad \text{--- (2)}$$

∵ We know that arc length

$$s = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^y \sqrt{1+\frac{y^2}{4a^2}}\,dy$$

$$= \frac{1}{2a}\int_0^y \sqrt{4a^2+y^2}\,dy$$

We know $\int \log\sqrt{a^2+x^2} = \frac{x}{2}$ $\quad \cancel{\frac{1}{2a}\left[x\right.}$

$$s = \frac{1}{2a}\left[\frac{1}{2}y\sqrt{4a^2+y^2} + \frac{1}{2}4a^2\log y+\sqrt{4a^2+y^2}\right]_0^y$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2\log\left(y+\sqrt{4a^2+y^2}\right) - 4a^2\log 2a\right]$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2\log\frac{y+\sqrt{4a^2+y^2}}{2a}\right]$$

From (2)

$$s = \frac{1}{4a}\left[2a\cot\psi\sqrt{4a^2+4a^2\cot^2\psi} + 4a^2\log\frac{2a\cot\psi+\sqrt{4a^2+4a^2\cot^2\psi}}{2a}\right]$$

$$= \frac{1}{4a}\left[2a\cot\psi\cdot 2a\,\mathrm{cosec}\,\psi + 4a^2\log\cot\psi+\mathrm{cosec}\,\psi\right]$$

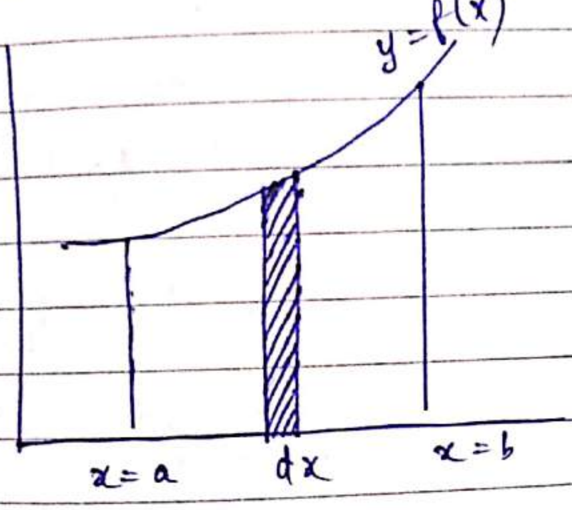
$$= a\cot\psi\,\mathrm{cosec}\,\psi + a\log(\cot\psi+\mathrm{cosec}\,\psi)$$

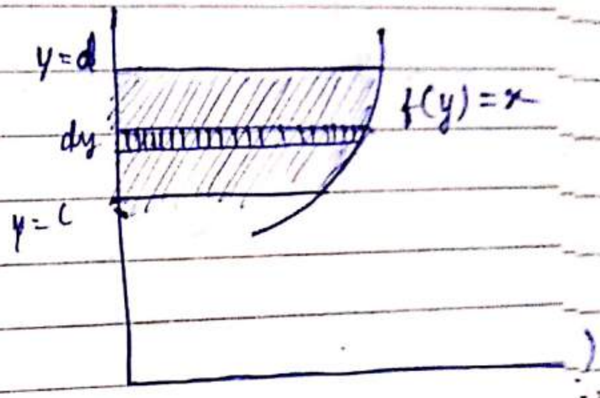
Quadrature

Quadrature is a method to find the area of region which is bounded by curves.

Area bounded by curve in cartesian eq. and coordinate axes:-

The area bounded by curve $y = f(x)$ the axis x and the ordinates $x = a$ and $x = b$ is





Area $\int_c^d f(y)\,dy$

Note

If curve eq is in parametric form then

$$\text{Area} = \int_a^b y\,dx$$

$$= \int_{t_1}^{t_2} y \frac{dx}{dt}\,dt$$

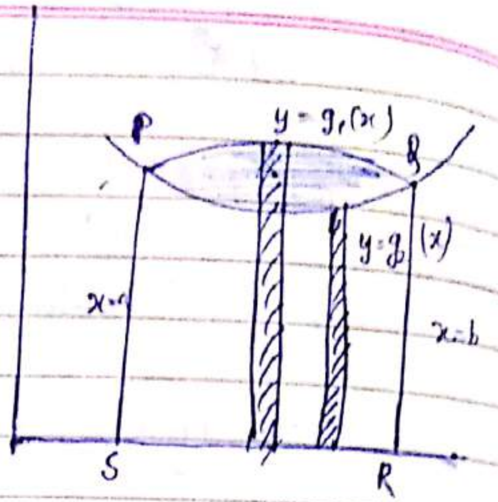
$$\text{Area} = \int_c^d x\,dx$$

$$= \int_{t_1}^{t_2} x \frac{dy}{dt}\,dt$$

Area Bounded By 2 Cartesian Curves

Area bounded by the curves

$y = g_1(x)$ and $y = g_2(x)$



= Area PAQRSP − area PBQRSP

$= \int_a^b g_1(x)\,dx - \int_a^b g_2(x)\,dx$

$= \int_a^b g_1(x) - g_2(x)\,dx$

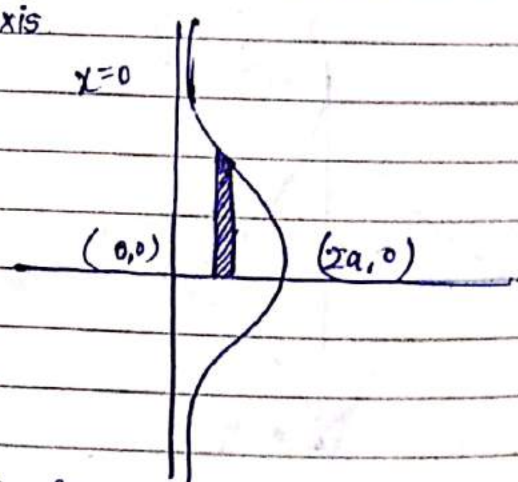
$= \int_a^b g_1(x) - g_2(x)\,dx$

Q Prove that the area bounded by the curve $xy^2 = 4a^2(2a-x)$ and its asymptotes is $4\pi a^2$

Given curve $xy^2 = 4a^2(2a-x)$ ——— ①

After curve tracing the shape of the curve ① is

Here asymptote is $x=0$ i.e. y axis



~~Req. area =~~

Area bounded by curve & its asymptotes $= 2\int_0^{2a} y\,dx$

From ①

$y = \pm 2a\sqrt{\frac{2a-x}{x}}$

Since curve lie in I quadrant, so y will be +ve

$= 2.2a \int_0^{2a} \sqrt{\frac{2a-x}{x}}\,dx$

Put $x = 2a\sin^2\theta$ $\quad x \to 0 \quad \theta \to 0$

$dx = 2a(2\sin\theta\cos\theta)\,d\theta$ $\quad x \to 2a \quad \theta \to \pi/2$

=

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$$= 4a^2 \int_0^{\pi/2} \frac{\sqrt{2}a}{\sqrt{2}a} \frac{\cos\theta}{\sin\theta} \, 4a \sin\theta \cos\theta \, d\theta$$

$$= 16a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$\because \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{16a^2}{2} \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

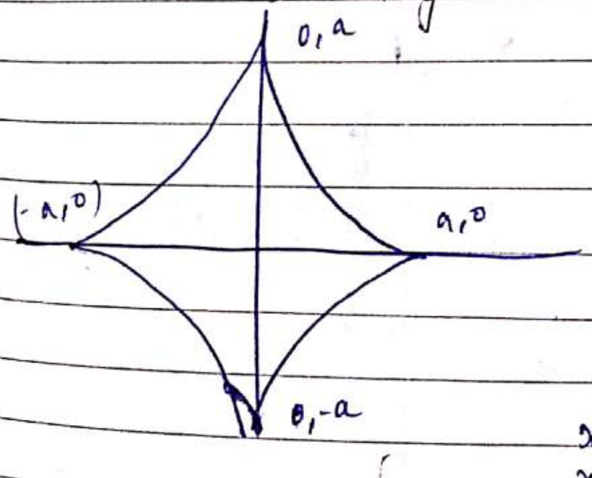
$$= \frac{16a^2}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{\Gamma 2}$$

$$= \frac{8a^2\pi}{2}$$

$$= 4\pi a^2$$

Ex. Prove that whole area of the curve

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } \frac{3}{8}\pi a^2$$



$$A = 4 \times \int_0^a y \, dx$$

$$A = 4 \times \int_0^a y \, dx$$

Put $x = \cos^3 t$ $\qquad$ $\frac{dx}{dt} = -3\cos^2 t \sin t$

$x \to 0 \qquad t \to \pi/2$

$x \to 1 \qquad t \to 0$

$$A = 4 \times \int_{\pi/2}^{0} a^2(\sin^3 t)(-3\cos^2 t \sin t) \, dt$$

$$= 12 a^2 \int_0^{\pi/2} \sin^4 t \cos^2 t \, dt = 12a^2 B\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{12a^2}{2} \frac{\Gamma 5/2 \, \Gamma 3/2}{\Gamma 4} = \frac{3}{8}\pi a^2$$

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Ex. 7 Find the area common to the following Tangents

$y^2 = ax$ $\qquad x^2+y^2 = 4ax$

$y^2 = ax$ $\qquad x^2 - 4ax + (2a)^2 - (2a)^2 + y^2 = 0$

$(x-2a)^2 + y^2 = (2a)^2$

$(x-2a)^2 + y^2 = 4a^2$

Symmetry :- $\because f(x,-y) = f(x,y)$ and $g(x,-y) = g(x,y)$

$\therefore$ curve is symmetric about X axis.

Origin :- $\because f(0,0) = 0$

$\therefore$ curve ~~does not~~ passes through origin.

Asymptotes: for $g(x,y)$

oblique asymptotes $\qquad y = m\ x = 1$

$\phi_3(m) = m^2+1 \qquad m = \pm i$

asymptote does not exist.

for $f(x,y)$, oblique asymptote

$\phi_2(m) = m^2$

$\phi_1(m) = -a$

$c = \dfrac{-\phi_1(m)}{\phi_2'(m)} = \dfrac{a}{2m}$

for $m = 0 \qquad c \to \infty$

so $y \to \infty$

so asymptote doesn't exist.

Intersection pts :- ~~for $f(x,y)$ when $y=0 \Rightarrow x=0$~~

~~for $g(x,y)$ when $y=0 \Rightarrow x(x-4a)=0$~~

~~$\Rightarrow x = 0, 4a$~~

$y^2 = ax \qquad x^2+y^2 = 4ax$

$x^2 = 3ax$

$x(x-3a) = 0$

$x = 0, \quad x = 3a$

$\Rightarrow \quad y = 0 \qquad x = \pm\sqrt{3}\,a$

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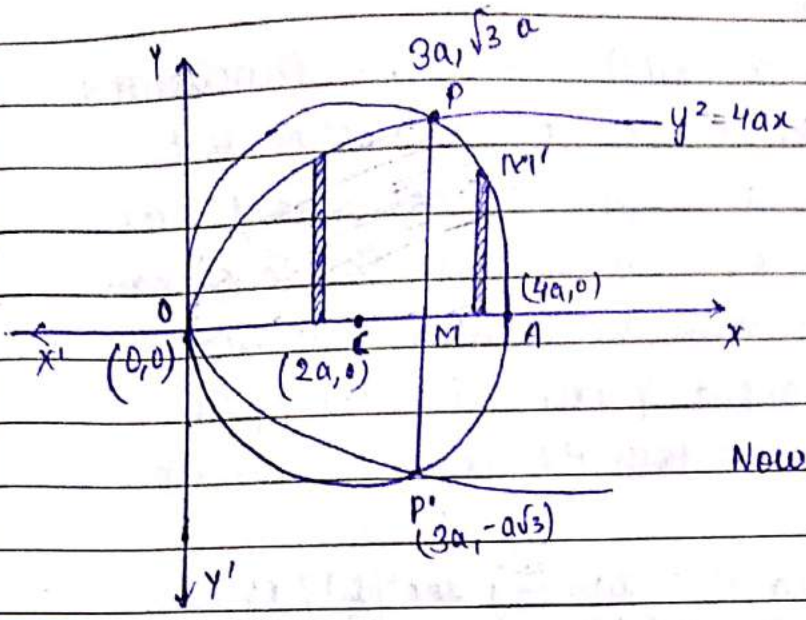
Region $y = \pm\sqrt{ax}$ $\quad x \geq 0$ $\qquad y^2 = 4ax - x^2$

$$y = \pm\sqrt{(4a-x)x}$$



$-$ 0 $+$ 4a $-$

Req. area $= 2 \times \int_0^{3a}$ area of OPCM dx $+ 2 \times \int_{3a}^{4a}$ area $PM'AM \, dx$

Now area of OPCM $= \int_0^{3a} \sqrt{ax}\, dx$

$$= \sqrt{a}\int_0^{3a} x^{1/2}\, dx$$

$$I_1 = \sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{3a} = 2\sqrt{3}a^2$$

Area of $PM'AM = \int_{3a}^{4a} \sqrt{(4a-x)x}\; dx$

$$I_2 = \int_{3a}^{4a} \sqrt{4a^2 - (x-2a)^2}\; dx$$

$$I_2 = \left[\frac{1}{2}(x-2a)\sqrt{4a^2-(x-2a)^2} + \frac{4a^2}{2}\sin^{-1}\left(\frac{x-2a}{2a}\right)\right]_{3a}^{4a}$$

$$= 2a^2 \sin^{-1} 1 - \left(\frac{a^2\sqrt{3}}{2} + 2a^2 \sin^{-1}\frac{1}{2}\right)$$

$$I_2 = 2a^2\frac{\pi}{2} - \frac{a^2\sqrt{3}}{2} - 2a^2\frac{\pi}{6}$$

$$\text{Req area} = 2\left[2\sqrt{3}a^2 + a^2\pi - \frac{\pi a^2}{3} - \frac{a^2\sqrt{3}}{2}\right]$$

$$= 2\left(\frac{2\pi a^2}{3} + \frac{3\sqrt{3}}{2}a^2\right)$$

$$= \frac{a^2}{3}\left[(4\pi + 9\sqrt{3})\right] \quad \text{Ans.}$$

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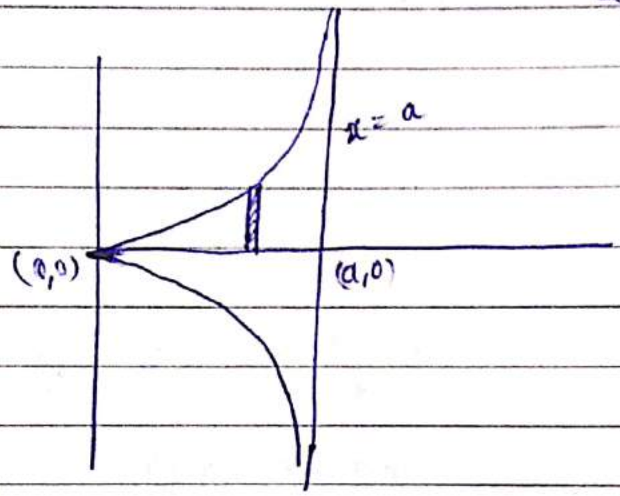
Q Find the area bounded by the curves $y^2(a-x) = x^3$ and its asymptotes.

$$y^2(a-x) = x^3$$

$$ay^2 - \quad a-x = 0$$

$$x = a \quad \text{parallel asymptotes parallel to } y \text{ axis}$$



Required area $= 2\int_0^a y\,dx$

$$= 2\int_0^a \sqrt{\frac{x^3}{a-x}}\,dx$$

Let $x = a\sin^2\theta$

$a - x = a\cos^2\theta$

Also $dx = 2a\sin\theta\cos\theta\,d\theta$

$x \to 0 \quad \theta \to 0$

$x \to a \quad \theta \to \pi/2$

Required area $= 2\int_0^{\pi/2} \frac{\sqrt{a^3\sin^6\theta}}{\sqrt{a}\cos\theta}\,2a\sin\theta\cos\theta\,d\theta$

$$= 4a^2\int_0^{\pi/2} \frac{\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \sin^4\theta\cos^0\theta\,d\theta$$

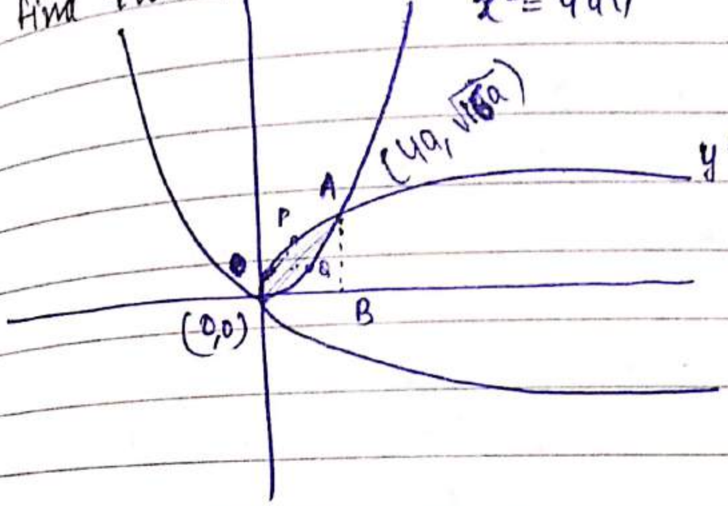
$$= \frac{4a^2}{2}\beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= \frac{4a^2}{2}\cdot\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= 2a^2\,\frac{\frac{3}{2}\times\frac{1}{2}\,\Gamma\frac{1}{2}\,\sqrt{\pi}}{2}$$

$$= \frac{3}{4}a^2\pi$$

Find the area included b/w the curve $y^2 = 4ax$ & $x^2 = 4ay$



Intersecting pt

$y^2 = 4ax \qquad x^2 = 4ay$

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$x^4 = 64a^3x$$

$$x(x^3 - 64a^3) = 0$$

$$x(x - 4a)(x^2 + 16 + 4x) = 0$$

$$x = 0, \quad x = 4a$$

$$y = 0 \quad y = \sqrt{16a} = 4\sqrt{a}$$

Intersecting pt is $(0,0)$ and $(4a, 4\sqrt{a})$

$$\text{Req area} = \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx$$

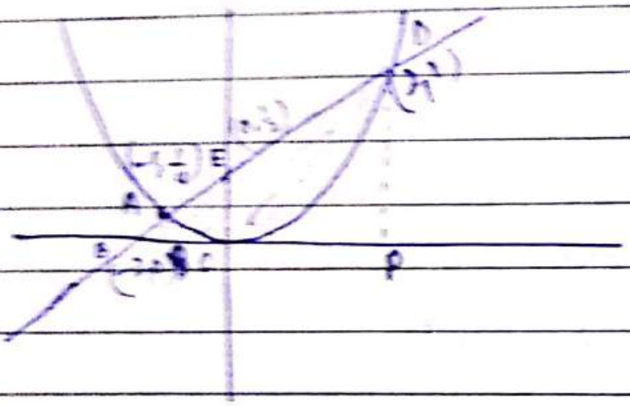
$$= 2\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{4a} - \frac{1}{12a}\left[x^3\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[8a\sqrt{a}\right] - \left[\frac{64a^3}{12a}\right]$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

MY ROUGH NOTE BOOK

Q. Prove that the ~~area~~ area b/w the parabola $x^2 = 4y$ & the line $x = 4y - 2$ is $\frac{9}{8}$.



$x = 4y - 2$

x	0	-2
y	$1/2$	0

Intersection pts. $x^2 = 4y$ $\quad x = 4y - 2$

$(4y-2)^2 = 4y$

$\Rightarrow 16y^2 + 4 - 16y = 4y$

$\Rightarrow 4y^2 - 5y + 1 = 0$

$\Rightarrow 4y^2 - 4y - y + 1 = 0$

$\Rightarrow (4y-1)(y-1) = 0$

$y = 1 \quad , 1/4$

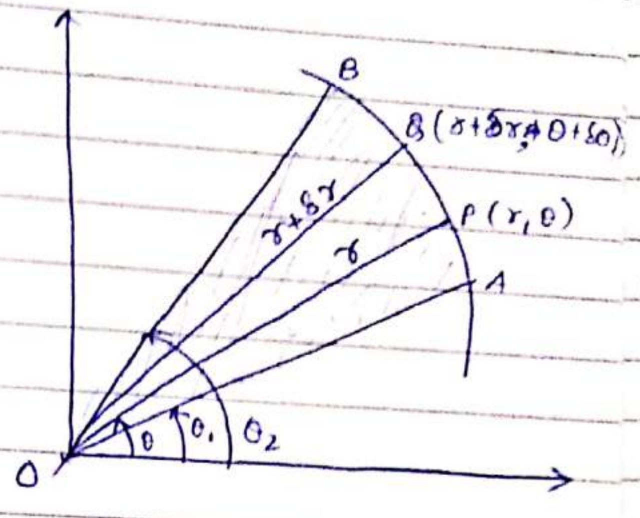
$x = 2 \quad -1$

Req. area $\int_{-1}^{2} \frac{x+2}{4} - \int_{-1}^{2} \frac{x^2}{4}$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{12}\left[x^3\right]_{-1}^{2} = \frac{1}{4}\left[2 + 4 - \frac{1}{2} + 2\right] = \frac{9}{8}$ sq. unit

Area Bounded By Curve in Polar ~~Form~~ equation and Radial Vectors.

Let curve eq. $r = f(\theta)$
and θ moves from θ_1 to θ_2



Area bounded by curve i.e

area OAB $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$

Area of closed curves

Area of closed curve are

$x = f_1(t)$ and $y = f_2(t)$

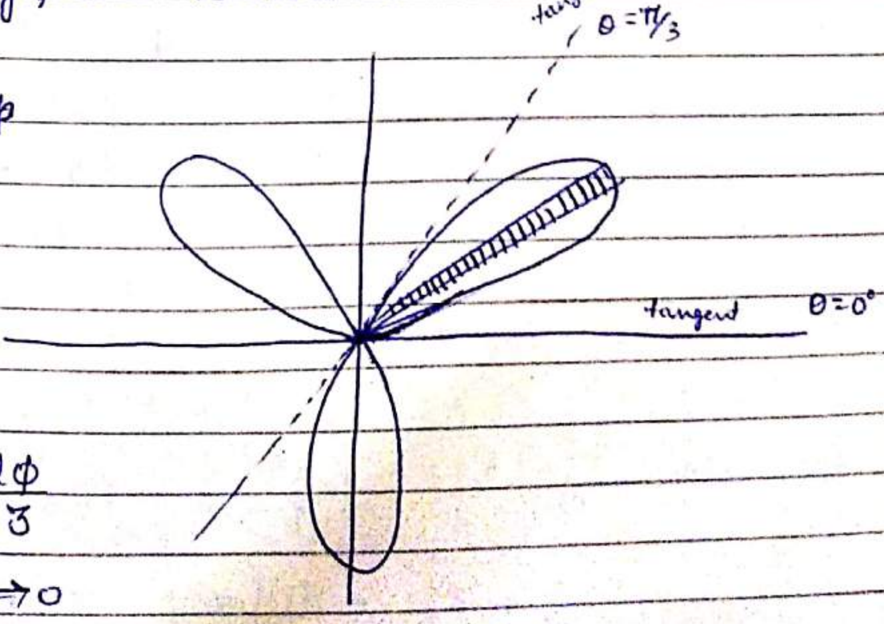
and t moves from t_1 to t_2

$\therefore$ Area $= \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$

Q Prove that the area of ~~one~~ whole loop at the curve $r = a\sin 3\theta$ is $\frac{1}{4}\pi a^2$.

Curve eq. $r = a\sin 3\theta$

After curve tracing, the curve is



Therefore area of one loop

$I = \frac{1}{2}\int_0^{\pi/3} r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} a^2\sin^2 3\theta\, d\theta$

$=$ Put $3\theta = \phi$

$d\theta = \frac{d\phi}{3}$

$\theta \to 0 \qquad \phi \to 0$

$\theta \to \pi/3 \qquad \phi \to \pi$

$$I = \frac{a^2}{2}\int_0^{\pi} \frac{\sin^2}{3}\phi \, d\phi$$

$$\because \{ f(a-x) = f(x)$$

$$\int_0^a f(x) = 2\int_0^{a/2} f(x)\,dx$$

$$\because \sin(\pi-\phi) = \sin\phi$$

$$I = \frac{a^2}{2}\times\frac{2}{3}\int_0^{\pi/2} \sin^2\phi \, d\phi$$

$$= \frac{a^2}{3}\int_0^{\pi/2} \sin^2\phi \cos^0\phi \qquad \int_0^{\pi/2} \sin^m\theta \cos^n\theta = \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{a^2}{3}\beta\left(\frac{2+1}{2}, \frac{1}{2}\right)$$

$$= \frac{a^2}{3}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} \qquad \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$= \frac{a^2}{3}\frac{\frac{1}{2}\pi}{2!}$$

$$I = \frac{\pi a^2}{12}$$

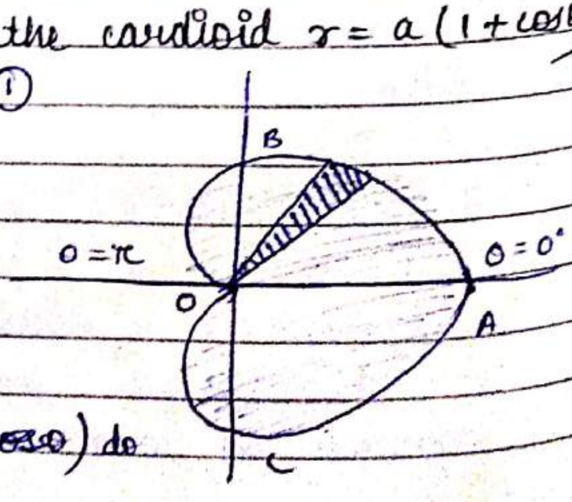
similarly area of 3 loop together $= 3I$

$$= \frac{\pi a^2}{4}$$

Q. Find the area enclosed by the cardioid $r = a(1+\cos\theta)$

Given curve $r = a(1+\cos\theta)$ —— ①

After curve tracing the obtained fig as.



$$\text{Area of } OAB = \frac{1}{2}\int_0^{\pi} r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos^2\theta)^2 \cancel{+2\cos\theta})\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left[2\cos^2\left(\frac{\theta}{2}\right)\right]^2 d\theta$$

$$I = \frac{a^2}{2}\times 4\int_0^{\pi}\cos^4\left(\frac{\theta}{2}\right) d\theta$$

Let $\frac{\theta}{2} = \phi$ $\quad \theta \to 0 \quad \phi = 0$

$\theta \to \pi \quad \phi \to \pi/2$

$d\theta = 2\,d\phi$

$$I = 2a^2\int_0^{\pi/2} 2\cos^4\phi\, d\phi$$

$$= \frac{4a^2}{2}\,\beta\left(\frac{1}{2},\frac{5}{2}\right) \qquad \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$= \frac{4a^2}{2}\,\frac{\Gamma\frac{1}{2}\;\frac{3}{2}\times\frac{1}{2}\;\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{4a^2}{\cancel{2}\times 2\times 2}\,\pi\times\frac{3}{4}$$

$$I = \frac{3\pi a^2}{4}$$

Area of the cardioid $= 2I$

$$= \frac{3\pi a^2}{2}$$

Q Prove that the area bounded by the curve $x = a\cos t$, $y = b\sin t$ is πab.

Given curve eq. $\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right\} — (1)$

at $t = 0 \Rightarrow x = a \quad y = 0$

$t = 2\pi \Rightarrow x = a \quad y = 0$

Since curve (1) is closed

$\Rightarrow t$ varies from 0 to 2π

Therefore ~~req.~~ area between the curves

$$= \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\ dt$$

$$= \frac{ab}{2}\left[t\right]_0^{2\pi}$$

$\sin^2 t + \cos^2 t = 1$

$\int_0^{2\pi} dt = [t]_0^{2\pi}$

$$= \pi ab$$

Q Prove that the area of the loop at the curve

$x = a(1-t^2)$ $\quad -1 \le t \le 1$ is $\frac{8}{15}a^2$

$y = at(1-t^2)$

$x = a(1-t^2)$ $\qquad y = at(1-t^2)$

at $t = -1$ $\qquad x = \cancel{2a}\ 0$ $\qquad y = \cancel{2at}\ 0$

at $t = 1$ $\qquad y = 0$ $\qquad y = 0$

$\therefore$ curve is closed

Area of the loop $= \frac{1}{2}\int_{-1}^{1} a(1-t^2)\left[a(1-t^2) + at(-2t)\right] - \left[at(1-t^2)(-2ta)\right] dt$

$$= \frac{1}{2}\int_{-1}^{1} a(1-t^2)\left[a - at^2 - 2at^2 + 2t^2a\right] dt$$

$$= \frac{1}{2}a^2\int_{-1}^{1}(1-t^2)^2\ dt$$

$\cancel{(1-t)^2 = 1 + t^2 - 2t}$

$$= \cancel{\frac{1}{2}a^2\left[\frac{t^3}{3} + t - \frac{t^2}{2}\right]_{-1}^{1}}$$

$$= \cancel{\frac{1}{2}a^2}$$

$$= \frac{1}{2}a^2\int_{-1}^{1} 1 + t^4 - 2t^2\ dt$$

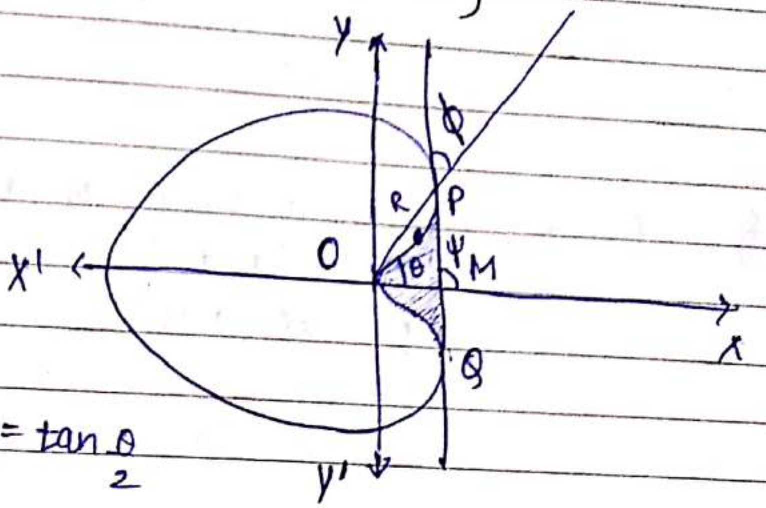
$$= \frac{1}{2}a^2\left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right]_{-1}^{1} = \frac{1}{2}a^2\left(\frac{16}{15}\right)$$

$$= \frac{8}{15}a^2$$

Q. Find the area included b/w the cardiod $r = a(1-\cos\theta)$ and its double tangents.

$\rightarrow \frac{a^2}{16}\left[15\sqrt{3} - 8\pi\right]$

PQ is double ~~pt~~ tangent of the cardioid which is ⊥ to X axis



$\therefore$ at P, $\psi = \pi/2$

$r = a(1-\cos\theta)$

$\frac{dr}{d\theta} = a\sin\theta$

$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \tan\frac{\theta}{2}$

$\phi = \frac{\theta}{2}$

$\because \theta + \phi = \psi$

$\frac{3\theta}{2} = \psi \Rightarrow \frac{3\theta}{2} = \frac{\pi}{2} \Rightarrow 3\theta = \pi$

$\theta = \pi/3$

$OP = a\left(1-\cos\frac{\pi}{3}\right) = \frac{a}{2}$

$\frac{OM}{OP}\ \cos\frac{\pi}{3} \Rightarrow OM = \frac{OP}{2} = \frac{a}{4}$

$MP = OP\sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}a$

$\therefore$ area of $\Delta\, OMP = \frac{1}{2}\, OM.\, MP = \frac{1}{2} \times \frac{\sqrt{3}a^2}{16} = \frac{\sqrt{3}a^2}{32}$

Now area of ORPM $= \frac{1}{2}\int_0^{\pi/3}\{a(1-\cos\theta)\}^3\, d\theta$

$= \frac{a^2}{2}\int_0^{\pi/3} 1 + \cos^2\theta.\ 2\cos\theta\, d\theta$

$$= \frac{a^2}{2}\left[\left(\theta - 2\sin\theta\right)_0^{\pi/3} + \int_0^{\pi/3} \frac{1+\cos 2\theta}{2}\, d\theta\right]$$

$$= a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{1}\right) + a\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/3}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{3} - \sqrt{3} + \frac{\pi}{6} + \frac{\sqrt{3}}{8}\right.$$

$$= \frac{a^2}{16}\left(4\pi - 7\sqrt{3}\right)$$

$\therefore$ Area enclosed by the curve about the initial line & double tangent $= 2\left(\Delta OMP - \text{ar } ORPM\right)$

$$= 2\left(\frac{\sqrt{3}a^2}{32} - \frac{a^2}{16}\left(4\pi - 7\sqrt{3}\right)\right)$$

$$= \frac{2a^2}{16}\left(\frac{\sqrt{3}}{2} - 4\pi + \frac{14}{2}\sqrt{3}\right)$$

$$= \frac{a^2}{16}\left(15\sqrt{3} - 8\pi\right)$$

$f'(x) > 0$ strictly ↑
Decreasing condition $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

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4. Mean Value Theorems

ROLLE'S THEOREM

If a function f defined on $[a, b]$ is
a) continuous on $[a, b]$
b) differentiable on (a, b)
c) $f(a) = f(b)$
then $\exists\ c \in (a, b)$ s.t. $f'(c) = 0$

Proof: $\because$ $f(x)$ is continuous in $[a, b]$. Therefore $f(x)$ is bounded and has supremum and infimum in $[a, b]$

Let supremum $f(x) = M$ and infimum $f(x) = m$
then $\exists$ $c, d \in [a, b]$ such that
$f(c) = m$ and $f(d) = M$

Here two possibilities

Case I: If $m = M$
then $f(x)$ is constant in $[a, b]$
$\therefore f'(x) = 0 \quad \forall\ x \in [a, b]$
$\Rightarrow f'(c) = 0 \quad \forall\ c \in [a, b]$

Case II: If $M \neq m$
then M, m can not be equal to $f(a)$ together
Therefore atleast one of them say m will be different from $f(a)$ or $f(b)$.
$\therefore f(c) = m \neq f(a) \Rightarrow c \neq a$ as $f(a) = f(b)$
$f(c) = m \neq f(b) \Rightarrow c \neq b$
$\therefore$ c lies in open interval (a, b).
Now we have to show that $f'(c) = 0$
Let, if possible $f'(c) \neq 0$ then there are two possibilities
(i) if $f'(c) < 0$
then $\exists\ \delta_1 > 0$ such that function $f(x)$ is decreasing in $(c, c + \delta_1)$

$\therefore \quad f(x) < f(c) = m$ $\quad$ [Becoz, $x \in (c, c+\delta_1)$ $\Rightarrow x > c \Rightarrow f(x) < f(c)$]

$\Rightarrow \quad f(x) < f(c) = m \qquad \forall\, x \in (c, c+\delta_1)$

$\Rightarrow \quad f(x) < m \qquad \forall\, x \in (c, c+\delta_1)$

which is contradiction to our fact that m is Infimum.

(ii) if $f'(c) > 0$

then $\exists\ \delta_2 > 0$ such that $f(x)$ is increasing in open interval $(c-\delta_2, c)$

$\therefore \quad f(x) < f(c)$ $\quad$ becz $x \in (c-\delta_2, c)$ $\Rightarrow x < c \Rightarrow f(x) < f(c)$

$\Rightarrow f(x) < f(c) = m$

$\Rightarrow f(x) < m$

which is again contradiction to our fact that m is Infimum.

Therefore our supposition is wrong.

Thus, $\exists\ c \in \cancel{[a,b]}\ (a,b)$

such that $\quad f'(c) = 0$

Algebraic Interpretation of Rolle's Theorem

If $x = a$ and $x = b$ are the two roots of a polynomial eq. $f(x) = 0$ then there lies atleast one root of eq. $f'(x) = 0$ between a and b.

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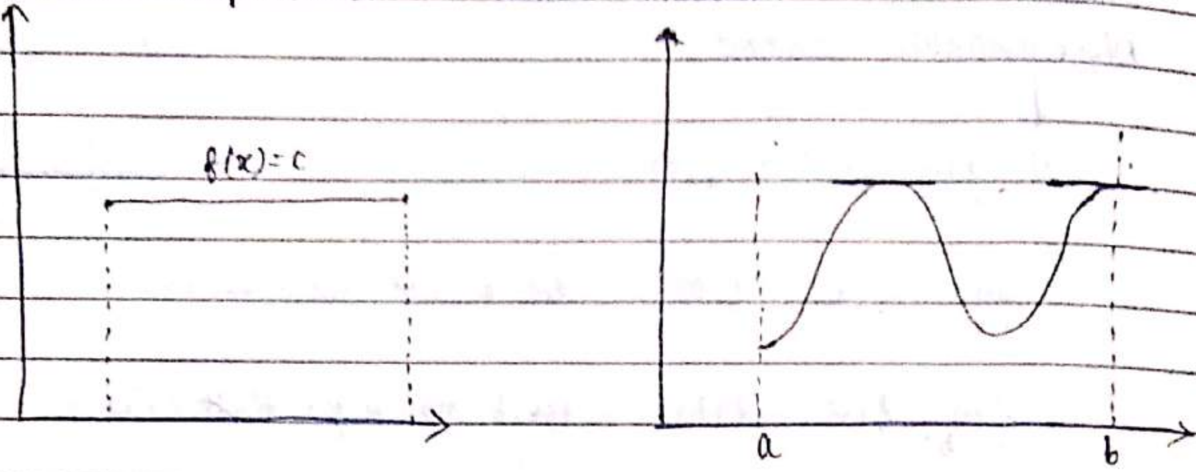
Geometrical Interpretation of Rolle's Theorem

Statement :-



If order of a curve at a point A and B are same & tangent at each point on the curve exist. Then atleast one pt. ~~from~~ b/w A and B where tangent is parallel to X axis.

Ex ⇒ Examine the validity of the hypothesis & conclusion of Rolle's Theorem for the foll

\# The converse is not true. Even if the function f does not satisfy the conditions of Rolle's theorem $f'(x) = 0$ at the pt. c in (a,b) i.e. for any $x \in (a,b)$, $f'(x) = 0$, all the three conditions of Rolle's Theorem are sufficient but not necessary.

Another Useful Form of Rolle's Theorem

If a function f ~~f~~ defined on $[a, a+h]$ is such that it is :

[R_1] continuous in $[a, a+h]$

[R_2] differentiable in $(a, a+h)$ and

[R_3] $f(a) = f(a+h)$

then $\exists\ \theta \in (0,1)$ such that $f'(a+\theta h) = 0$

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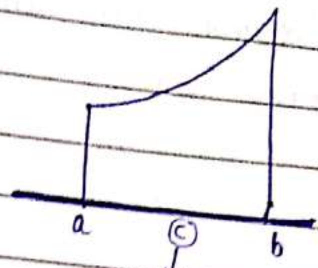
Continuous नहीं break न होगा

Differentiable smooth

↓

if $\lim_{x \to c} f(x) = f(c)$



$\lim_{x \to a^+} f(x) = f(a)$ for a no left limit exist

arbitrary

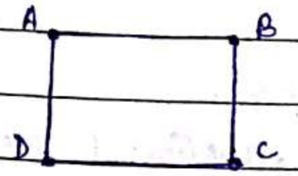
$\lim_{x \to b^-} f(x) = f(b)$ for b no right limit exist

It will be differentiable when following limit exist

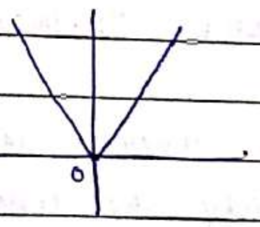
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or Put $x = a + h$ as $x \to a$ $h \to 0$

$$\therefore \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Continuous but not differentiable at A, B, C, D



Continuous but not differentiable at O.

Close intervals means finite values of end points.

Bounded function - A function $f(x)$ is said to be bounded if

$\exists$ $M > 0$ s.t. $|f(x)| = M$

or $\exists$ m, M s.t. $m \le f(x) \le M$

lower bound

upper bound

(1) Bounded above (2) Bounded below

let $Z^- = \{-1, -2, -3, \ldots\}$

$x \leq -1 \quad \forall x \in Z^-$

So, -1 is bounded above set

$Z^+ = \{1, 2, 3, 4, \ldots\}$

This set is bounded below set.

Similar concept applied in case of function.

Bounded above function :- A function $f(x)$ is said to be bounded above if $\exists \, M \in R$ such that

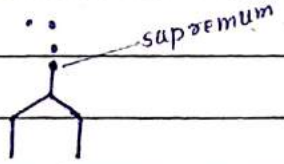
$$f(x) \leq M \quad \forall x$$

upper bound

Bounded below function :- A function $f(x)$ is said to be bounded below if $\exists \, m \in R$ such that

$$f(x) \geq m \quad \forall x$$

lower bound



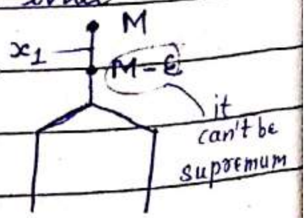
SUPREMUM (least upper bound)

let $f : A \to B$ be a real valued function then $M \in R$ is said to be supremum of function $f(x)$ if

(i) $f(x) \leq M \quad \forall x \in A$

ii) $\forall \, \varepsilon > 0$ (however small) $\exists \, x_1 \in A$ such that

$$f(x_1) > M - \varepsilon$$



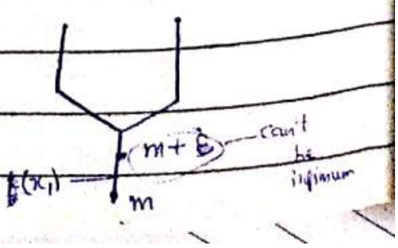
INFIMUM (greatest ~~upper~~ lower bound)

Let $f : A \to B$ be a real valued function then $m \in R$

i) $f(x) \geq m \quad \forall x \in A$

ii) $\forall \, \varepsilon > 0$ (however small) $\exists \, x_1 \in A$ s.t.

$$f(x_1) < m + \varepsilon$$



There are infinite upper bound

$x \in \{-1, -2, -3, \ldots\}$

$x \leq -1$ supremum

$x \leq -2$

$x \leq +3$

$\left.\begin{matrix} x \leq -1 \\ x \leq -2 \\ x \leq +3 \end{matrix}\right\}$ infinite upper bound.

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Ex-2 Is Rolle's theorem applicable for the following functions in the interval mentioned against them. If so, then verify Rolle's theorem:

a) $f(x) = e^x \sin x$ $[0, \pi]$

b) $f(x) = |x|$ $[-1, 1]$

a) R_1 Since the standard functions e^x & $\sin x$ are continuous for every value of x, therefore their product will also be continuous for every value of x, particularly, in $[0, \pi]$

R_2 $f'(x) = e^x(\sin x + \cos x)$ which is not infinite or indeterminate, therefore the function f is differentiable in $(0, \pi)$

R_3 $f(0) = e^0 \sin 0 = 0$ $\sin 0 = 0$

$f(\pi) = e^\pi \sin \pi = 0$ $\sin \pi = 0$

$\therefore f(0) = f(\pi)$

Thus the given function f satisfies all the 3 conditions of Rolle's Theorem. Therefore, Rolle's theorem is applicable & accordingly there must be atleast one point $c \in (0, \pi)$, where $f'(c) = 0$

$\Rightarrow e^c(\sin c + \cos c) = 0$

$\because$ since for any finite value of c, $e^c \neq 0$

$\therefore \sin c + \cos c = 0 \Rightarrow \tan c = -1$

$\Rightarrow \tan c = \tan(-\pi/4)$

$\Rightarrow c = n\pi + (-\pi/4)$ $n \in Z$

$$\boxed{\begin{array}{l} \text{If } \tan\theta = \tan\alpha \\ \text{then } \theta = n\pi + \alpha \end{array}}$$ $n \in Z$

Replacing $n = 0, 1, 2, 3, \ldots$

$c = -\pi/4, 3\pi/4, 7\pi/4, 11\pi/4, \ldots$

Clearly, $\frac{3\pi}{4} \in (0, \pi)$

Every Polynomial is Differentiable & Continuous on Real Line.

$\log_a b = \dfrac{\log_e b}{\log_e a}$

$\log_{a^m} b^n = \dfrac{n}{m} \log_a b$

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b) $f(x) = |x|$ $\quad [-1, 1]$

R_1 $f(x) = |x|$ is continuous for every value of x, particularly in $[-1, 1]$

$f(x) = \begin{cases} -x & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$

R_2 $f(x)$ is not differentiable at $x = 0$

$R f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} =$

$L f'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x} = -$

Therefore the function f is not differentiable in $(-1, 1)$

Hence Rolle's theorem is not applicable for the given function in $(-1, 1)$

Ex-3 Is Rolle's theorem applicable for the following function in the interval $[a, b]$? If yes, then verify the theorem:

$$f(x) = \log\left\{\frac{x^2 + ab}{x(a+b)}\right\}, \qquad 0 \in [a, b]$$

continuous $\quad$ polynomial is continuous

R_1 $f(x) = \log(x^2 + ab) - \log x - \log(a+b)$ is continuous in $[a, b]$ being composite function of continuous functions in $[a, b]$

R_2: $f'(x) = \dfrac{2x}{x^2 + ab} - \dfrac{1}{x}$ is not infinite or indeterminate for $a < x < b$.

$0 \notin (a,b)$ so it verified

therefore $f'(x)$ is differentiable in (a, b).

R_3: $f(a) = \log\left\{\dfrac{a^2 + ab}{a(a+b)}\right\} = 0$

$f(b) = \log\left(\dfrac{b^2 + ab}{b(a+b)}\right) = \log 1 = 0$

Therefore $f(a) = f(b)$

Thus, the given function f satisfies all the three conditions of Rolle's theorem. Therefore, Rolle's theorem is applicable and ~~so~~ accordingly there must be atleast one pt. $x = c$ in (a, b) where $f'(c) = 0$

i.e. $f'(c) = \dfrac{2c}{c^2 + ab} - \dfrac{1}{c} = 0$

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$$c^2 = ab$$
$$c = \pm\sqrt{ab}$$

a, b, c in G/P then $\frac{b}{a} = \frac{c}{b}$
$b^2 = ac$
$b = \sqrt{ac}$

Out of these two value of c, one $\sqrt{ab} \in (a,b)$, being G.M of a and b and $-\sqrt{ab} \notin (a,b)$
Hence Rolle's theorem is applicable for $f(x)$ on $[a, b]$

Ex-4 Show that b/w any 2 roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x = 1$.

Let a and b be the roots of eq. $e^x \cos x = 1$, then

$$\left.\begin{aligned} e^a \cos a = 1 \\ e^b \cos b = 1 \end{aligned}\right\} \quad \text{—— (1)}$$

Consider the following function defined in $[a, b]$

$$e^x \cos x = 1$$
$$\cos x = \frac{1}{e^x} = e^{-x}$$

$$f(x) = e^{-x} - \cos x \quad \text{—— (2)}$$

R_1 The given function being the difference of 2 standard continuous functions in $[a, b]$ is continuous in $[a, b]$.

R_2 $f'(x) = -e^{-x} + \sin x$ exists for $x \in (a, b)$
$\therefore$ $f(x)$ is differentiable in (a, b)

R_3 $f(a) = e^{-a} - \cos a = e^{-a} - e^{-a} = 0$
$f(b) = e^{-b} - \cos b = e^{-b} - e^{-b} = 0$ (from (1))
~~Hence~~ $\therefore$ $f(a) = f(b)$

Thus the given function f satisfies all the 3 conditions of Rolle's theorem. Therefore, Rolle's theorem is applicable and accordingly there must be atleast one point c in (a, b) where $f'(c) = 0$

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$\Rightarrow \quad -e^{-c} + \sin c = 0$

$\Rightarrow \quad e^{c} \sin c - 1 = 0$

$\Rightarrow$ the roots of $e^{x} \sin x - 1 = 0$ is $c \in (a, b)$

Hence, atleast one root of $e^{x} \sin x = 1$ lies b/w 2 roots of $e^{x} \cos x = 1$

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First Mean Value Theorem

Lagrange's Mean Value Theorem

If a function f with domain $[a, b]$ is such that it is

(L_1) continuous in $[a, b]$, and

(L_2) differentiable in (a, b)

then $\exists\, c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Let us define a new function F with domain $[a, b]$ involving the given function as follows:

$$F(x) = f(x) + kx$$

where ~~A~~ k is a constant ~~to~~ be determined such that

$$F(a) = F(b)$$

$$f(a) + ka = f(b) + kb$$

$$f(a) - f(b) = k(b - a)$$

Now show that $k = f'(c)$

$$k = -\frac{[f(b) - f(a)]}{b - a} \qquad (1)$$

R_1 R_2 $\because$ $f(x)$ & kx are continuous in closed interval $[a, b]$ and differentiable in (a, b). So sum of 2 continuous functions on $[a, b]$ and differentiable in (a, b)

R_3 $F(a) = F(b)$ [by the condition of constant k]

Thus, F satisfies all the 3 conditions of Rolle's Theorem therefore accordingly, ~~therefore~~ must be atleast one point c in (a, b) such that

$$F'(c) = 0$$

$$f'(c) + k = 0$$

$$\Rightarrow f'(c) = -k$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \{\text{from } (1)\}$$

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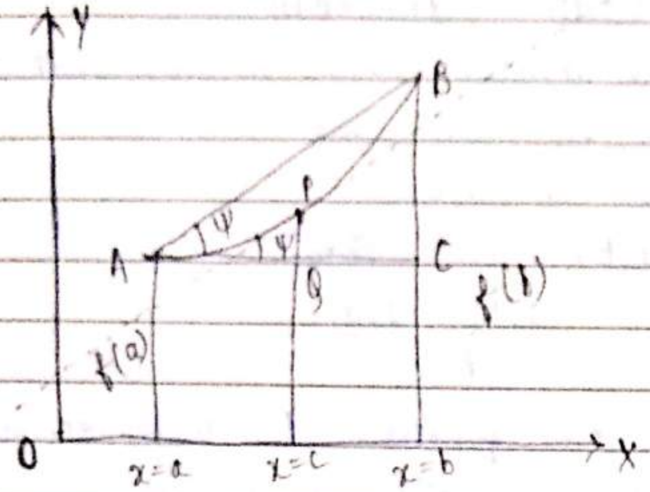
Geometrical Meaning of Mean Value Theorem



If a curve $y=f(x)$ is continuous b/w two given points whose abscissae are $x=a$ and $x=b$ respectively & a tangent can be drawn to the curve at every point then there exists atleast one point $x=c$, $c \in (a,b)$ such that the tangents there at is parallel to the chord joining the two given end points.

~~tan ψ=~~

→ Let the arc APB represents the graph of the function $y=f(x)$ and a and b be the ordinates of A and B respectively join AB. Draw perpendiculars from A and B on x axis. Let the chord AB makes an angle ψ with the x-axis, then from the right angled Δ ACB.

$$\tan\psi = \frac{CB}{AC}$$

$$= \frac{f(b)-f(a)}{b-a} = f'(c) \qquad a<c<b$$

(from Lagrange's MVT)

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ANOTHER USEFUL FORM OF LAGRANGE'S MEAN VALUE THEOREM.

If a function $f(x)$ defined on close interval $[a, a+h]$ such that it is

i) continuous on $[a, a+h]$ and

ii) differentiable on $(a, a+h)$,

then there is atleast one point θ lies in open interval $(0,1)$ such that

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$$

If we put $b-a=h$ in Lagrange's mean value theorem
$b = a+h$

then any pt. c can be taken as $c = a+\theta h$ between a and b
where $0 < \theta < 1$

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{a+h-a}$$

$$f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$$

Important Deductions from Lagrange's MVT

If a function $f(x)$ is continuous in interval $[a,b]$ & differentiable in interval (a,b) then $\forall\ x \in (a,b)$.

a) $f'(x) = 0 \Rightarrow f(x)$ is constant in interval $[a,b]$.

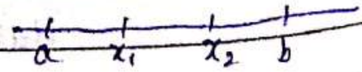
b) $f'(x) < 0 \Rightarrow f(x)$ is strictly decreasing in interval $[a,b]$

c) $f'(x) > 0 \Rightarrow f(x)$ is strictly increasing in interval $[a,b]$.

Let x_1 and x_2 $(x_1 < x_2)$ be any two diff. pts in (a,b)

i.e. $a < x_1 < x_2 < b$

$\Rightarrow [x_1, x_2] \subset [a,b]$



If $[a,b]$ is continuous & differentiable in (a,b)

$\Rightarrow$ continuous in $[x_1, x_2]$
differentiable in (x_1, x_2)

By Lagrange's Theorem $\exists\ c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{—— ①}$$

a) $f'(x) = 0 \qquad \forall\ x \in (a,b)$

Put $x = c$

$f'(c) = 0 \qquad \forall\ c \in (a,b)$

from eq. ①

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \qquad \forall\ x_1, x_2 \in (a,b)$$

$\Rightarrow f(x_2) = f(x_1) \qquad \forall\ x_1, x_2 \in (a,b)$

$\Rightarrow f(x)$ is constant in $[a,b]$

b) $f'(x) < 0 \qquad \forall\ x\ (a,b)$

Put $x = c$

$f'(c) < 0$

from ① $\quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$

$f(x_2) < f(x_1)$

But $x_2 < x_1$

thus f is ~~monotonically~~ strictly decreasing in $[a,b]$

c) $f'(x) > 0 \qquad \forall\ x \in (a,b)$

Put $x = c$

$f'(c) > 0$

from ①, $\quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \qquad \forall\ x_1, x_2 \in (a,b)$

$f(x_2) > f(x_1)$

But $\quad x_2 > x_1$

Thus f is strictly increasing in $[a,b]$

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Theorem If two functions $f(x)$ and $g(x)$ are continuous in closed interval $[a,b]$, differentiable in open interval (a,b) and $f'(x) = g'(x)$ $\forall x \in (a,b)$, then the difference of the functions $f(x) - g(x)$ is constant in closed interval $[a,b]$

Define a new function $F(x) = f(x) - g(x)$

$$F'(x) = f'(x) - g'(x) = 0 \qquad [\because f'(x) = g'(x)]$$

$\because$ $f(x)$ and $g(x)$ are continuous in $[a,b]$ & differentiable in (a,b). Thus $F(x)$ are continuous $[a,b]$ & differentiable in (a,b).

$$\Rightarrow F'(x) = 0$$

$$\Rightarrow F(x) = \text{constant function.}$$

$$\Rightarrow f(x) - g(x) = \text{constant in } [a,b]$$

Extra

Example- Apply Lagrange's MVT for function $f(x) = \log(1+x)$ to prove that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$ $\forall x > 0$

For $\forall$ $x > 0$, $f(x) = \log(1+x)$ is continuous in $[0,x]$ & differentiable in $(0,x)$

$$f'(x) = \frac{1}{1+x}$$

We know $f'(a+\theta h) = \frac{f(a+h) - f(a)}{h}$ $\quad [a, a+h]$

$$f'(0+\theta x) = \frac{f(0+x) - f(0)}{x}$$

$[0, x]$
$\Rightarrow [0, 0+x]$
$0 < \theta < 1$

$$f'(\theta x) = \frac{f(x) - f(0)}{x}$$

so, $a = 0$
$h = x$

$$x f'(\theta x) = f(x) \qquad f(0) = \log 1 = 0$$

$$\frac{x}{1+\theta x} = \log(x+1) \qquad 0 < \theta < 1$$

Again $x > 0$ & $0 < \theta < 1$

$$\Rightarrow 0 < \theta x < x$$

$$\Rightarrow 1 < 1 + \theta x < 1 + x$$

$0 < \theta < 1$
let $x = 1/4$ $\quad \theta = 1/2$
$\Rightarrow \theta x = 1/8$
$\Rightarrow \theta x < x$

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$$\Rightarrow \frac{1}{1+x} < \frac{1}{1+\theta x} < 1$$

$$\Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x} < x$$

$$\frac{x}{1+x} < \log(1+x) < x$$

$$\frac{1}{x} < [\log(1+x)]^{-1} < \frac{x+1}{x}$$

$$\frac{1}{x} - \frac{1}{x} < [\log(1+x)]^{-1} - x^{-1} < \frac{x+1}{x} - \frac{1}{x}$$

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1$$

extra
Example 2 Verify Lagrange's MVT for the following fn:
$f(x) = x(x-1)(x-2) \quad \forall x \in [0, 1/2]$

x, $(x-1)$, $(x-2)$ are continuous & differentiable, so $f(x)$ will also be continuous in $[0, 1/2]$ & differentiable in $(0, 1/2)$ ∴ satisfying Lagrange's MVT. So acc. to theorem ∃ at least one c in $(0, 1/2)$ where

$$f'(c) = \frac{f(1/2) - f(0)}{1/2 - 0}$$

$$(c-1)(c-2) + c(c-2) + c(c-1) = \frac{3/8 - 0}{1/2} = \frac{3}{4}$$

$$(c^2 - 3c + 2) + c^2 - 2c + c^2 - c = 3/4$$

$$3c^2 - 6c + 2 = 3/4$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24}$$

$$c = 1 \pm \frac{\sqrt{21}}{6}$$

Here $c = 1 + \sqrt{21}/6 > 1/2$ but $c = 1 - \sqrt{21}/6 < 1/2$

$$c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)$$

Hence verified LMVT

CAUCHY'S MEAN VALUE THEOREM

If in a closed interval $[a,b]$ two function $f(x)$ & $g(x)$ are defined such that both functions are

i) continuous in closed interval $[a,b]$

ii) differentiable in open interval (a,b) &

iii) $f'(x) \neq 0$ & $\forall\ x \in (a,b)$

then $\exists\, c \in (a,b)$ s.t. $\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{g'(c)}{f'(c)}$

Proof: first show that the ~~func~~ $\frac{g(b) - g(a)}{f(b) - f(a)}$ is defined.

for this show that $f(b) \neq f a$

so let $f(a) = f(b)$

then satisfies all conditions of Rolle's theorem.

Hence $\exists\ c \in (a,b)$ s.t. $f'(c) \neq 0$ but this contradicts with 3^{rd} condition of our given theorem.

Hence $f(a) \neq f(b)$

Again assume $\phi(x) = g(x) + A f(x)$ $\forall\, x \in (a,b)$

where A is const. s.t. $\phi(a) = \phi(b)$

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i.e. $g(a) + A f(a) = g(b) + A f(b)$ ——— (1)

$$A = -\frac{g(b) - g(a)}{f(b) - f(a)}$$

Here A exists as $f(a) \neq f(b)$

Since $\phi(x)$ is sum of two continuous & differentiable function

Thus ϕ is continuous in $[a,b]$ & differentiable in (a,b)

Thus function $\phi(x)$ satisfies all conditions of Rolle's Theorem.

So, $\exists\ c \in (a,b)$ s.t.

$$\phi'(c) = 0$$

$$g'(c) + A f'(c) = 0$$

$$A = -\frac{g'(c)}{f'(c)} \quad \text{——— (2)}$$

From (1) & (2),

$$\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{g'(c)}{f'(c)}$$

If we take $f(x) = x$ in Cauchy's MVT then we'll get Lagrange's MVT.

ANOTHER FORM OF CAUCHY'S THEOREM.

If in interval $[a, a+h]$ two functions $f(x)$ & $g(x)$ are defined s.t. both functions are

i) continuous in interval $[a, a+h]$

ii) differentiable in interval $(a, a+h)$

& $f'(x) \neq 0 \quad \forall x \in (a, a+h)$

then $\exists\ \theta \in (0,1)$ s.t.

$$\frac{g(a+h) - g(a)}{f(a+h) - f(a)} = \frac{g'(a+\theta h)}{f'(a+\theta h)}$$

~~E~~Example 1) Verify Cauchy's MVT for the following function:
$f(x) = x^2 \quad g(x) = x^3, \ \forall x \in [1,2]$

$f(x)$ & $g(x)$ are continuous in $[1,2]$ & differentiable in $(1,2)$ since $f(x)$ & $g(x)$ are continuous and differentiable in all intervals.
Also $g(x) = x^2 \neq 0 \quad \forall x (1,2) \quad$ since $0 \notin (1,2)$

Thus satisfy all 3 conditions of Cauchy's MVT.
$\exists c \in (1,2)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

$$\frac{2c}{3c^2} = \frac{4-1}{8-1} = \frac{3}{7}$$

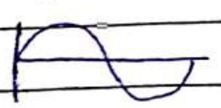
$$\frac{2c}{3c^2} = \frac{3}{7}$$

$$14c = 9c^2$$

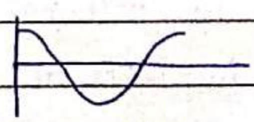
$\cancel{c \in \frac{14}{9}}$ $\quad c(9c - 14) = 0$
$c = 0, \ c = 14/9$

$c = 0 \notin (1,2)$
but $c = \frac{14}{9} \in (1,2)$

~~E~~ Ex-2 Find value of θ using Cauchy's MVT if $f(x) = \sin x$ & $g(x) = \cos x$, $\forall x \in [0, \pi/2]$



sin x



cos x

no break = continuous
no corner pt; smooth = differentiable

$\boxed{\text{Differentiability}}$
$f(x) = \sin x$
$f'(x) = \cos x$

Now check for given interval $[0, \pi/2]$ if it has finite value it is differentiable
$\cos 0 = 1 \qquad \cos \pi/2 = 0 \qquad$ so $f(x) = \sin x$ is differentiable

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$f(x)$ & $g(x)$ are continuous & differentiable

$g(x) = \cos x$ in (a, b)

$g'(x) = -\sin x \neq 0$ in $(0, \pi/2)$

Applying Cauchy's MVT

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

$$\frac{\sin(a+h) - \sin a}{\cos(a+h) - \cos a} = -\frac{\cos(a+\theta h)}{\sin(a+\theta h)}$$

$$= \frac{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{-2\sin\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)} = -\cot(a+\theta h)$$

$$= \frac{2\cos\left(\frac{2a+h}{2}\right)\sin\frac{h}{2}}{-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)} = -\cot(a+\theta h)$$

$$= \cot\left(\frac{2a+h}{2}\right) = \cos(a+\theta h)$$

$$a + \frac{h}{2} = a + \theta h$$

$$\theta = 1/2 \in (0, 1)$$

Ex 3 Use Cauchy's MVT to evaluate the following

$$\lim_{x \to 1}\left[\frac{\cos(\pi x/2)}{\log(1/x)}\right]$$

$f(x) = \cos(\pi x/2)$ $\quad g(x) = \log\left(\frac{1}{x}\right)$

$a = x$ $\quad b = 1$

By Cauchy's MVT $\quad \frac{f(1) - f(x)}{g(1) - g(x)} = \frac{f'(c)}{g'(c)}$ $\quad x < c < 1$

$$\frac{\cos \pi/2 - \cos \pi x/2}{\log 1 - \log 1/x} = \frac{-\pi/2 \sin(\pi c/2)}{\frac{-c}{c^2}}$$

$$\Rightarrow \frac{\cos \pi x/2}{\log(1/x)} = \frac{\pi}{2} c \sin\left(\frac{\pi c}{2}\right) \quad \text{——} (1)$$

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if $x \to 1$ $\qquad x < c < 1$
$\Rightarrow c \to 1$

then from ①

$$\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{\log\left(\frac{1}{x}\right)} = \lim_{c\to 1} \frac{\pi c}{2} \sin\left(\frac{\pi c}{2}\right)$$

$$= \frac{\pi}{2} \sin\frac{\pi}{2} = \frac{\pi}{2}$$

Q Examine the validity of the hypothesis & conclusions of Rolle's Theorem for following function.

a) $f(x) = 1 - |x-1|$, $x \in [0,2]$

b) $f(x) = x^3 - 4x$, $x \in [-2,2]$

a) $|x-1|$

So first $x - 1 = 0 \Rightarrow x = 1$

$$|x-1| = \begin{cases} -(x-1) & x \le 1 \\ x-1 & x > 1 \end{cases}$$

equal कहीं भी लगाओ या दोनों जगह लगाओ

$$\therefore \; 1 - |x-1| = \begin{cases} 1+x-1 = x & 0 \le x \le 1 \\ 1-x+1 = 2-x & 1 < x \le 2 \end{cases}$$

To check differentiability

R $\quad f'(1) = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^+} \frac{2-x-1}{x-1}$

Put $x = 1+h$

$$= \lim_{h\to 0} \frac{2-1-h-1}{1+h-1} = \lim_{h\to 0} \frac{-h}{h} = -1$$

LHL $\quad f'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x-1}{x-1}$

$x = 1-h$ at $h = 0$

$$= \lim_{h\to 0} \frac{\cancel{1}-h-\cancel{1}}{\cancel{1}-h-\cancel{1}} = \lim_{h\to 0} \frac{-h}{-h} = 1$$

Here $Rf'(1) \neq Lf'(1)$ and $1 \in [0,2]$ & given f^n is not differentiable at $x=1$
Thus, Rolle's Theorem is not verified for ~~int~~ ~~fo~~ given function in interval $[0,2]$.

b) $f(x) = x^3 - 4x$ $\quad x \in [-2,2]$

Since polynomial is continuous & differentiable ~~in~~ on real no.
it is also continuous in $[-2,2]$ & differentiable in $(-2,2)$

And $f(-2) = -8+8 = 0$
$f(2) = 8-8 = 0$
i.e. $f(-2) = f(2)$

Thus, given function satisfies all 3 conditions of Rolle's Theorem

Now $f'(x) = 3x^2 - 4 = 0$
$x^2 = 4/3$
$x = \pm 2/\sqrt{3}$

$\frac{2}{\sqrt{3}} \in (-2,2)$

Thus, in interval $(-2,2)$ there exists atleast one pt $c = \frac{2}{\sqrt{3}} \in (-2,2)$ where $f'(c) = 0$. Hence Rolle's Theorem is verified

Ex. If $f(x) = (x-5)\log x$, then show that the eq. $x\log x + x - 5 = 0$ is satisfied by atleast one value lying b/w 1 & 5.

$f(x) = (x-5)\log x$ continuous in $[1,5]$

$f'(x) = \log x + \frac{x-5}{x}$

also $f'(x)$ is finite in $(1,5)$. Thus $f(x)$ is differentiable in interval $(1,5)$

& $f(1) = -4\log 1 = 0$
$f(5) = (5-5)\log 5 = 0$

$\Rightarrow f(1) = f(5)$

$\therefore$ satisfies all 3 condition of Rolle's theorem. Thus $\exists$ exist a pt in $(1,5)$ s.t. $f'(x) = 0 \Rightarrow \log x + \frac{x-5}{x} = 0 \Rightarrow x\log x + x - 5 = 0$

Thus eq $x\log x + x - 5 = 0$ interval $(1,5)$ में x के कम से कम एक मान के लिए संतुष्ट है।

GENERAL MEAN VALUE THEOREM

If in interval [a,b] three functions $f(x)$, $g(x)$ & $h(x)$ defined in such a way that all 3 functions are
i) continuous in interval [a,b] &
ii) differentiable in interval (a,b)
then $\exists c \in (a,b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Also deduce the cauchy's MVT & Lagrange's MVT using by this theorem.

Proof: Let $F(x)$ be a function in closed interval [a,b] such that

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$= f(x)\underbrace{[g(a)h(b) - h(a)g(b)]}_{\text{constant}} + g(x) - - -$$

$$= A f(x) + B g(x) + C h(x) \quad \xrightarrow{(\text{Let})} \text{①}$$

where A, B, C are constants
we know $f(x), g(x), h(x)$ are continuous in [a,b] and differentiable in (a,b). Thus linear combination of differentiable f^n is again differentiable & linear combination of continuous f^n is again continuous.
Thus $F(x)$ is continuous in [a,b] & differentiable in (a,b).
Also $\quad F(a) = 0 \quad F(b) = 0$

$$\Rightarrow F(a) = F(b)$$

Thus $F(x)$ satisfy all 3 condition of Rolle's Theorem
So, $\exists\ c \in (a,b)$ s.t.

$$F'(x) = 0$$

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$$\Rightarrow \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Deduction:

i) If $h(x) = k$ then

$$h(a) = h(b) = k$$

because $h(x) = k$
तो x की value कुछ भी रखो
k ही आएगी।
$h(a) = k$
$h(b) = k$

so, $h'(x) = 0$

$$h'(c) = 0 \qquad \forall\ c \in (a,b)$$

Thus from General Mean Value Theorem

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & k \\ f(b) & g(b) & k \end{vmatrix} = 0$$

$$\Rightarrow f'(c)\left[g(a)k - g(b)\,k\right] - g'(c)\left[k\,f(a) - k\,f(b)\right] = 0$$

$$\Rightarrow f'(c)\left[g(a) - g(b)\right] = g'(c)\left[f(a) - f(b)\right]$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This is Cauchy's Mean Value Theorem

ii) If $g(x) = x$ & $h(x) = k$ where k is constant then, by general mean value theorem

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & k \\ f(b) & b & k \end{vmatrix} = 0$$

$\because g(x) = x$
$\Rightarrow g(a) = a$
$\&\ g(b) = b$

$$\Rightarrow k f'(c)\left[a - b\right] = 1 . k\left[f(a) - f(b)\right]$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

which is Lagrange's Mean Value Theorem

⇒ $f(x)$ differentiable है तो $f'(x)$ exist करेगा
⇒ If $f'(x)$ exist then $f(x)$ is differentiable

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SECOND MEAN VALUE THEOREM
द्वितीय माध्यमान प्रमेय

If function $f(x)$ defined in closed interval $[a,b]$ in such a way that $f'(x)$ is
i) exists & continuous in closed interval $[a,b]$.
ii) differentiable in open interval (a,b)
then $\exists\, c \in (a,b)$ s.t.
$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(c)$$

Proof: Let

polynomial

$$g(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} B \quad \text{——} ①$$

so

continuous & differentiable

where B is constant such that $g(a) = g(b)$
$$g(a) = g(b)$$
$$f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} B = f(b) \quad \text{——} ②$$

Thus $g(x)$ is differentiable & continuous. since linear combination of differentiable & continuous f^n is differentiable & continuous. Also $g(a) = g(b)$.
Hence, all 3 conditions of Rolle's Theorem is satisfied.
Then there exists $c \in (a,b)$ ——③
$$g'(c) = 0$$
but
$$g'(x) = f'(x) + (b-x) f''(x) - f'(x) - (b-x) B$$
$$g'(x) = (b-x) f''(x) - (b-x) B$$
$$g'(c) = (b-c) f''(c) - (b-c) B$$
[from ③]
$$(b-c)[f''(c) - B] = 0$$
$$B = f''(c)$$

$\because (b-c) \neq 0$
$\because c \in$ open interval (a,b)
$\therefore c \neq a$ & $c \neq b$
i.e. $b-c \neq 0$ as b is not equal to c

a c b

From (2)

$$f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(c) = f(b)$$

If a function $f(x)$ with domain $[a, a+h]$ then $\exists\, \theta \in (0,1)$ for which Second Mean Value Theorem will be as-

$$\boxed{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a+\theta h)}$$

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Example 6 If in interval $[a,b]$ function $f(x)$ is continuous & possess finite derivatives for $x = c \in (a,b)$ then show that

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}$$

General MVT के लिए 3 function की जरूरत होती है, so we will use Second MVT

$f'(x)$ at $x = c$ exist

Thus, $f'(x)$ will exist in the neighbourhood of c $(c-h, c+h)$ $\exists\, h > 0$

$c-h \quad c \quad c+h$

$(c-h, c) \quad (c, c+h)$

for both interval, condition of Second MVT is applied

Now applying Second MVT in interval $(c-h, c)$ and $(c, c+h)$

$$f(c+h) = f(c) + h f'(c) + \frac{h^2}{2!} f''(c+\theta_1 h) \qquad 0 < \theta_1 < 1 \quad \text{(1)}$$

replace h by $-h$ in eq. (1)

$$f(c-h) = f(c) - h f'(c) + \frac{h^2}{2!} f''(c-\theta_2 h) \qquad 0 < \theta_2 < 1 \quad \text{(2)}$$

On adding eq. (1) & (2)

$$f(c+h) + f(c-h) = 2f(c) + \frac{h^2}{2!}\left[f''(c+\theta_1 h) + f''(c-\theta_2 h)\right]$$

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = \lim_{h\to 0} \frac{1}{2!}\left[f''(c+\theta_1 h) + f''(c-\theta_2 h)\right]$$

Put $h \to 0$

so, $\frac{2}{2!} f''(c) = f''(c)$

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = f''(c)$$

$$\begin{array}{ccccccc} & & & 1 & & 1 & & \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \end{array}$$

$$(x+h)^3 = x^3h^0 + 3x^2h^1 + 3xh^2 + x^0h^3$$

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Ex-2. Find the value of θ if

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h)$$

where (i) $f(x) = x^3 + x$
(ii) $f(x) = (x-a)^{5/2}$ when $x \to a$

$$f(x) = x^3 + x \qquad f(x+h) = (x+h)^3 + x + h$$
$$f'(x) = 3x^2 + 1$$
$$f''(x) = 6x$$
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h)$$
$$(x+h)^3 + x + h = x^3 + x + h(3x^2+1) + \frac{h^2}{2!} 6(x+\theta h)$$
$$(x+h)^3 = x^3 + 3x^2h + 3h^2(x+\theta h)$$
$$\Rightarrow h^3 = 3\theta h^3$$
$$\Rightarrow \theta = \frac{1}{3} \in (0,1)$$

ii) $f(x) = (x-a)^{5/2}$
$$f'(x) = \frac{5}{2}(x-a)^{3/2}$$
$$f''(x) = \frac{15}{4}(x-a)^{1/2}$$
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h)$$
$$(x-a+h)^{5/2} = (x-a)^{5/2} + h\frac{5}{2}(x-a)^{3/2} + \frac{h^2}{2}\frac{15}{4}(x-a+\theta h)^{1/2}$$

On taking lim $x \to a$
$$\lim_{x\to a}(x-a+h)^{5/2} = \lim_{x\to a}(x-a)^{5/2} + \frac{5h}{2}(x-a)^{3/2} + \frac{15}{8}h^2(x-a+\theta h)^{1/2}$$
$$h^{5/2} = \frac{15}{8}h^2(\theta h)^{1/2} \quad \Rightarrow \quad \theta^{1/2} = \frac{8}{15}$$

$$\frac{f(a)}{b-a}$$

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If $f'(x)$ is finite
then $f(x)$ is differentiable.

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$\theta = \frac{64}{225}$

GENERALISED MEAN VALUE THEOREM

Taylor's Theorem with Lagrange's form of Remainder.

If function $f(x)$ defined in interval $[a, a+h]$ such that

i) derivatives $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ upto $(n-1)$ orders are continuous in interval $[a, a+h]$

ii) derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ upto n orders are exists in interval $(a, a+h)$ then $\exists\, \theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{(n-1)}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

PROOF:

Let us define a function $g(x)$

$$g(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{(a+h-x)^n}{n!} B \quad \text{—}①$$

let $[a, a+h]$ be $[x, a+h]$

where B is constant such that

$$g(a) = g(a+h)$$

$$f(a) + f'(a).h + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} B = f(a+h) \quad ②$$

Since $f(x), f'(x), \ldots f^{(n-1)}(x)$ is continuous in interval $[a, a+h]$ and differentiable in $(a, a+h)$. Also $(a+h-x), \frac{(a+h-x)^2}{2!}, \ldots, \frac{(a+h-x)^n}{n!}$ is polynomial which is also differentiable & continuous in real no.. And we know linear combination of product of 2 continuous & differentiable fn is again continuous &

$\exists\, c \in (a, a+h)$

$c = a+\theta h \quad \theta \in (0,1) \quad 0<\theta<1$

If $\theta = 0$, $c = a$
$\theta = 1$ $c = a+\theta h$ } i.e. lie in $(a, a+h)$

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differentiable, therefore $g(x)$ is continuous in $[a, a+h]$ and differentiable in $(a, a+h)$

and also $\quad g(a) = g(a+h)$

So all conditions of Rolle's theorem is satisfied. Thus, $\exists\, \theta \in (0,1)$ such that

$$g'(a+\theta h) = 0 \qquad \text{——— (3)}$$

So differentiate (1) w.r.t. x.

$$g'(x) = f'(x) + (a+h-x) f''(x) - f'(x) - \frac{2(a+h-x)}{2!} f''(x) + \frac{(a+h-x)^2}{2!} f'''(x) + \dots$$

$$- \cancel{\frac{(n-1)(a+h-x)^{n-2}}{(n-1)!}} f^{(n-1)}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x)$$

$$- \frac{n(a+h-x)^{(n-1)}}{n!} B$$

$$g'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} \left[f^{(n)}(x) - B \right] \qquad \text{——— (4)}$$

From eq. (3) & (4), we get
i.e. x को $a+\theta h$ से replace

$$g'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} \left[f^{(n)}(a+\theta h) - B \right] = 0$$

$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} \left[f^{(n)}(a+\theta h) - B \right] = 0$$

Also $h>0$

$[\because 1-\theta \neq 0]$
as $\theta \in (0,1)$
So, ~~$\theta \neq 0$ & $\theta \neq 1$~~
So, $1-\theta \neq 0$ as θ cannot have value 1

since $h > 0$ and $(1-\theta) \neq 0$ so,

$$\Rightarrow f^{(n)}(a+\theta h) = B$$

Put value of B in eq. (2)

$$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{(n-1)}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

$0 < \theta < 1$

Here,

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$ which is $(n+1)^{th}$ term. It is called LAGRANGE'S FORM OF REMAINDER जो h के आरोही पूर्णांकों घातों में $f(a+h)$ के टेलर प्रसार में n पदों के बाद शेष रहता है।

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Particular Cases :-

1. Put $c = a + (b-a)\theta$ and $h = b-a$ in Taylor's Theorem

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^n(c) \qquad c \in (a,b)$$

$c = a + \theta h$

2. Put $n = 1$ in Taylor's Theorem, we get Lagrange's MVT

$$f(b) = f(a) + (b-a) f'(a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

3 In above 1. let $h = x - a$ then

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{(x-a)^n}{n!} f^{(n)}(c) \qquad a < c < x$$

$\therefore c = a + \theta(x-a)$

इसके फलन द्वारा $f(x)$ का $(x-a)$ की घातों में प्रसार कर सकते हैं)

Taylor's Theorem with Cauchy's Form of Remainder

If function $f(x)$ defined in interval $[a, a+h]$ such that

i) derivatives $f'(x), f''(x), \dots, f^{(n-1)}(x)$ of up to $(n-1)$ order are continuous in interval $[a, a+h]$

ii) derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$ upto n order are exists in interval $(a, a+h)$ then $\exists\ \theta \in (0,1)$ such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{(n-1)}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$$

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Proof: Let

$$g(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x)$$

$$+ \dots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x)B \quad \text{--- (1)}$$

where B is constant such that

$$g(a) = g(a+h)$$

$$\Rightarrow f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ hB \quad \text{--- (2)}$$

$\because$ $f(x), f'(x), \dots, f^{(n-1)}(x)$ are differentiable in $(a, a+h)$

Also linear combination of product of 2 continuous and differentiable is continuous & differentiable

$\therefore$ $g(x)$ is continuous in $[a, a+h]$ & differentiable in $(a, a+h)$

Also $g(a+h) = g(a)$.

Thus satisfy all 3 conditions of Rolle's theorem. Then $\exists\ \theta \in (0,1)$ such that

$$g'(a+\theta h) = 0 \quad \text{--- (3)}$$

Differentiating eq. (1) wrt x,

$$g'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - f''(x).(a+h-x)$$

$$+ \frac{(a+h-x)}{2!} f'''(x) + \dots + -\frac{(n-1)[a+h-x]^{n-2}}{(n-1)!} f^{(n-1)}x$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - B$$

$$g'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - B \quad \text{--- (4)}$$

from (3) & (4) $\quad \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - B = 0$

$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - B = 0$$

$$B = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

Put value of B in (2)

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+ \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

Remainder After n terms:-

$$R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$$

Taylor's Theorem with Schlomitch & Roche Form of Remainder

If function $f(x)$ defined on interval $[a, a+h]$ such that

i) derivatives $f'(x), f''(x), \dots f^{(n-1)}$ upto $(n-1)$ orders are continuous on interval $[a, a+h]$

ii) derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$ upto n ~~terms~~ orders are exists on interval $(a, a+h)$ and $p \in N$; then $\exists\, \theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}a$$
$$+ \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h); \quad 0<\theta<1$$

PROOF:

Let $g(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \dots$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + (a+h-x)^p B \quad \text{—— (1)}$$

it is polynomial as $p \in N$ so p will be non integral +ve no.

where B is constant such that $g(a) = g(a+h)$

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + h^p B = f(a+h)$$
—— (2)

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Since $f(x), f'(x), \ldots, f^{(n-1)}(x)$ in interval $(a, a+h)$ is differentiable तथा x के बहुपद अवकलनीय होते हैं। (polynomials of x)

$\because$ Linear combination of continuous fns & differentiable function is again a continuous & differentiable function

Thus function $g(x)$ in interval $[a, a+h]$ is continuous & differentiable in $(a, a+h)$

Also $g(a) = g(a+h)$

Thus $g(x)$ satisfy all 3 conditions of Rolle's Theorem

Then $\exists\ \theta \in (0,1)$ such that

$$g'(a+\theta h) = 0 \quad \text{——— (3)}$$

$$g'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - p(a+h-x)^{p-1} B \quad \text{——— (4)}$$

From (3) & (4),

$$g'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^{n}(a+\theta h) - p(h-\theta h)^{p-1} B = 0$$

$$\Rightarrow \frac{h^{n-1}}{(n-1)!} (1-\theta)^{n-1} f^{n}(a+\theta h) = p\, h^{p-1} (1-\theta)^{p-1} B$$

$$\frac{h^{n-p}(1-\theta)^{n-p}}{p\,(n-1)!} f^{n}(a+\theta h) = B \quad \text{——— (5)}$$

Put value of B in (2),

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$$

Remainder after n terms

$$R_n = \frac{h^n (1-\theta)^{n-p}}{p\,(n-1)!} f^{(n)}(a+\theta h)$$

\# Put $p=n$ in above theorem

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

Taylor's theorem with Lagrange's form of Remainder.

$[a, a+h] = [0, x]$
$= [0, 0+x]$

PAGE NO. $a=0 \quad h=x$
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Put $p=1$ in above.

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n (a+\theta h)$$

Taylor's theorem with Cauchy's form of remainder.

MACLAURIN'S THEOREM

If function $f(x)$ defined on interval $[0, x]$ such that

i) derivatives upto $(n-1)$ orders are continuous on interval $[0, x]$

ii) derivatives upto n orders exist in interval $(0, x)$ and $p \in N$;

then $\exists \ \theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)} \theta + R_n$$

$$R_n = \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(\theta x),$$

is called Maclaurin's theorem with Schlomitch and Roche form of Remainder.

PROOF :- Same proof as that of Schlomitch & Roche form

Just put $a=0 \quad h=x$

In above theorem put $p=1$

we get Taylor's theorem with Cauchy's form of Remainder

If $p=n$ then $R_n = \frac{x^n}{n!} f^n(\theta x)$

Maclaurin's theorem in Lagrange's form of Remainder

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POWER SERIES (घात श्रेणी)

श्रेणी $\sum_{n=0}^{\infty} a_n (x-a)^n$ बिंदु $x = a$ पर घात श्रेणी कहलाती है

जहाँ a_n तथा a अचर है तथा x वास्तविक चर राशि है।

यदि $a=0$ हो तो श्रेणी $\sum_{n=0}^{\infty} a_n x^n$ जहाँ a_n अचर है, मानक घात श्रेणी कहलाती है।

[टेलर श्रेणी Taylor's series]

यदि अंतराल $[a, a+h]$ में फलन परिभाषित है तथा टेलर प्रमेय के सभी प्रतिबंधों को संतुष्ट करता है तो टेलर प्रमेयानुसार

$$f(a+h) = f(a) + h f'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

जहाँ R_n, n पदों के बाद टेलर का शेष कहलाता है।

Let function $f(x)$ in which each derivative in close interval $[a, a+h]$ exists and $R_n \to 0$ when $n \to \infty$ $\forall x \in [a, a+h]$ then $f(x)$ can be expand in infinite series form.

Thus,

$$f(a+h) = \lim_{n \to \infty} \left[f(a) + h f'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) \right]$$

$$\because R_n \to 0 \text{ as } n \to \infty$$

$$\therefore f(a+h) = f(a) + \frac{h}{1!} f'(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + \dots$$

अतः फलन का दक्षिण पक्ष (RHS) टेलर श्रेणी कहलाती है।

Maclaurin's series

$a = 0 \quad h = x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

Example 1) use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h \sin z \cdot \frac{\sin z}{1} - (h \sin z)^2 \frac{\sin(2z)}{2} + \dots$$

where $z = \cot^{-1} x$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) \quad \text{(1)}$$

$$f(x+h) = \tan^{-1}(x+h)$$

$$f(x) = \tan^{-1}(x)$$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \frac{1}{\operatorname{cosec}^2 z} = \sin^2 z$$

$$f'(x) = \sin^2 z$$

$$f''(x) = -\frac{1}{(1+x^2)^2}(2x) = -\frac{2\cot z}{(1+\cot^2 z)^2} = -2\cot z \sin^4 z$$

put this in (1)

$$\tan^{-1}(x+h) = \tan^{-1}x + h \sin^2 z + \frac{h^2}{2!}(-2\cot z \sin^4 z) + \dots$$

$$\Rightarrow \tan^{-1}(x+h) = \tan^{-1}x + h \sin z \cdot \frac{\sin z}{1} - (h \sin z)^2 \frac{2 \sin z \cos z}{2} + \dots$$

$$\Rightarrow \tan^{-1}(x+h) = \tan^{-1}x + (h \sin z)\frac{\sin(z)}{1} - (h \sin z)^2 \frac{\sin(2z)}{2} + \dots$$

Ex-2 Find Cauchy's & lagrange's remainder after n terms in the expansion of following function $\log(1+x)$

Let $f(x) = \log(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

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On Maclaurin's expansion of f(x)

Cauchy's Remainder

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$$

$$= \frac{x^n (1-\theta)^{n-1}}{\cancel{(n-1)!}} \cdot \frac{(-1)^{n-1} \cancel{(n-1)!}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \frac{x^n}{(1+\theta x)^n} (1-\theta)^{n-1}$$

$$= (-1)^{n-1} \frac{x^n}{(1+\theta x)} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

Lagrange's Remainder

$$R_n = \frac{x^n}{n!} f^n(\theta x)$$

$$= \frac{x^n}{n!} \frac{(-1)^{n-1} (n-1)!}{(1+\theta x)^n}$$

$$= \frac{x^n}{n} \frac{(-1)^{n-1}}{(1+\theta x)^n}$$

$$= \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x}\right)^n$$

Ex-3 Show that the function $f(x) = e^{1/x}$ can not be expanded in Maclaurin's Series

$$f(x) = e^{1/x}$$

$x = 0$ $f(x) = e^{\infty} = \infty$ infinite

$$f'(x) = -\frac{1}{x^2} e^{1/x}$$

$$f''(x) = e^{1/x} \left(\frac{2}{x^3} + \frac{1}{x^4}\right)$$

Here value of function & its derivative at $x = 0$ is infinite.

Thus, can't be expanded through Maclaurin's ~~theorem~~ Series

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Maclaurin's के लिए $R_n \to 0$ when $n \to \infty$

Q Find the power series expansion of $\sin x$

$$f(x) = \sin x$$
$$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$$
$$f''(x) = -\sin x = \sin\left(2 \cdot \frac{\pi}{2} + x\right)$$

$$f^n(x) = \sin\left(\frac{n\pi}{2} + x\right), \quad n \in N, \quad x \in R \qquad \text{--- (1)}$$

$\therefore$ in interval $[0, x]$ n^{th} derivative exist.

From (1) put $x = 0$

$$f^n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\left(\frac{n-1}{2}\right)} & \text{if } n \text{ is odd} \end{cases}$$

$\sin \pi/2 = 1$
$\sin 3\pi/2 = -1$
$\sin 5\pi/2 = 1$

अतः घात श्रेणी के प्रसार के लिए सिद्ध करना है कि $\lim_{n \to \infty} R_n(x) \to 0$. इसके लिए मैकलारिन प्रसार से

Lagrange's Remainder

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) \qquad 0 < \theta < 1$$

$$= \frac{x^n}{n!} \sin\left(\theta x + n\frac{\pi}{2}\right)$$

$$|R_n| = \left|\frac{x^n}{n!} \sin\left(\theta x + n\frac{\pi}{2}\right)\right|$$

$$\leq \left|\frac{x^n}{n!}\right| \qquad [\because |\sin x| \leq 1]$$

$$\lim_{n\to\infty} |R_n| \leq \lim_{n\to\infty} \left|\frac{x^n}{n!}\right|$$

$$= 0 \qquad \boxed{\because \Sigma \left|\frac{x^n}{n!}\right| \text{ is convergent serie}}$$

$$\lim_{n\to\infty} |R_n| = 0$$

Thus satisfying all conditions of Maclaurin's series. Thus,

$f(0) = n^{\theta} = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad \forall x \in R$$