By Cauchy's general principle of Convergence for Sequences, a necessary and Sufficient Condition for the Convergence of $\langle S_n \rangle$ is that given $\varepsilon > 0$, $\exists\, n_0 \in N$ such that

$$|S_n - S_m| < \varepsilon, \quad \forall\ n > m \geq n_0 \qquad \text{--- (ii)}$$

$\therefore$ (i) and (ii) $\Rightarrow$ $|u_{m+1} + u_{m+2} + \dots + u_n| < \varepsilon$

$\forall\ n > m \geq n_0$

Hence the theorem.

Thm A necessary (but not sufficient) Condition for the Convergence of a Series Σu_n is that $\lim u_n = 0$

i.e., Σu_n is Convergent $\Rightarrow \lim\limits_{n \to \infty} u_n = 0$

but Converse may not be true

eg: Let the Series $\Sigma \frac{1}{n}$

Here, $u_n = \frac{1}{n}$

so, $\lim\limits_{n\to\infty} u_n = 0$, but it is not Convergent Series.

Various important tests for Convergence of infinite series.

1. Comparison tests:

Let Σu_n and Σv_n be two series of Positive terms and n_0 be some fixed positive integer such that

* If $u_n \leq K v_n$ $\forall n \geq n_0$ where K is a fixed positive number, then

(i) Σu_n Converges if Σv_n Converges.

(ii) Σv_n diverges if Σu_n diverges.

* If $u_n \geq K' v_n$, $\forall n \geq n_0$
$K \rightarrow$ +ve number

Then
(i) Σu_n diverges if Σv_n diverges

(ii) Σv_n Converges if Σu_n Converges.

Theorem :- If Σu_n and Σv_n be two series of Positive terms, such that

$$\lim \frac{u_n}{v_n} = l \neq 0$$

then the two series Σu_n and Σv_n Converges or diverges Simultaneously.

Some important Fundamental Comparison Series.

1. [Geometric Series]

The series $\Sigma u_n = 1 + r + r^2 + \dots + r^{n-1} + r^n + \dots$

(i) Converges to $\frac{1}{1-r}$ if $|r| < 1$

(ii) diverges to ∞ (infinity) if $r \geq 1$

2. [Hyper harmonic Series or p-series]

The Series $\Sigma u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$

is Convergent if $p > 1$

& divergent if $p \leq 1$

Example :- Test the Convergency or divergency of the series

$$\frac{1}{2} + \frac{\sqrt{2}}{5} + \frac{\sqrt{3}}{8} + \dots + \frac{\sqrt{n}}{3n-1} + \dots$$

Sol^n

Here $u_n = \frac{\sqrt{n}}{3n-1} = \frac{\sqrt{n}}{3n(1-\frac{1}{3n})}$

$$= \frac{1}{3\sqrt{n}\,(1-1/3n)}$$

Hence take the auxiliary series $v_n = \frac{1}{\sqrt{n}}$

then

$$\frac{u_n}{v_n} = \frac{1}{3\sqrt{n}\,(1-1/3n)} \cdot \frac{\sqrt{n}}{1}$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$, which is non zero finite quantity

$\therefore$ Σu_n and Σv_n Cong. or div. Simultaneously.

Σv_n is divergent because $p = 1/2 < 1$ (by p-test)

So, given series is divergent.