[D' Alembert's Ratio Test]

Let Σu_n be a series of Positive terms Such that

(a) $\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}}\right) = l$

then (i) if $l > 1$, Σu_n is cgt.

(ii) if $l < 1$, Σu_n is divergent

(iii) if $l = 1$, Σu_n may cgt or diver

(b) If $\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}}\right) \to \infty$, then

Σu_n Converges.

eg: Test the Convergence of the series whose n^{th} term is $\frac{n^n}{n!}$

Sol^n Here $u_n = \frac{n^n}{n!}$, $u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

Now, $\frac{u_n}{u_{n+1}} = \frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{(n+1)}}$

$$= \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{(n+1)} \cdot \frac{(n+1)!}{n!}$$

$$= \frac{1}{(1+1/n)^n} \cdot \left(\frac{1}{n+1}\right) \cdot \frac{(n+1)}{1}$$

$$= \frac{1}{(1+1/n)^n}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

$\because \quad 2 < e < 3$

Therefore by ratio test the given series is divergent.

Cauchy's n^{th} root test :-

Let Σu_n be a series of Positive terms such that

$$\lim_{n \to \infty} (u_n)^{1/n} = l$$

(i) if $l < 1$, then Σu_n is Convergent

(ii) if $l > 1$, then Σu_n is divergent

(iii) if $l = 1$, Σu_n may Conv. or diverge.

eg: Let $u_n = \left(\frac{nx}{n+1}\right)^n$

$$\therefore \lim_{n \to \infty} \left\{\left(\frac{nx}{n+1}\right)^n\right\}^{1/n} = \lim_{n \to \infty} \left(\frac{nx}{n+1}\right)$$

$$= \underset{n \to \infty}{Lt} \left(\frac{x}{1 + 1/n}\right) = x$$

Hence, by Cauchy's n^{th} root test the Series is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$, then $u_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1 + 1/n)^n}$

$$\lim_{n \to \infty} u_n = \frac{1}{e} \neq 0$$

$\Rightarrow \Sigma u_n$ is divergent.