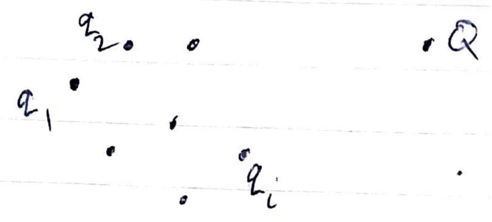
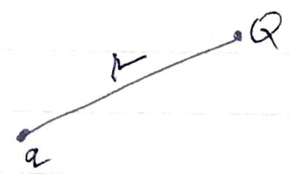
Electrostatics

The Electric field





Source charges "Test charge"

↓

Principle of superposition :- Interaction of two charges is completely unaffected by the presence of others.

$$F = F_1 + F_2 + F_3 + \cdots$$

Not only the position, velocity and acceleration of q right now that matter.

Electromagnetic news ⟶ travels at the speed of light

What concerns Q ⟶ position, velocity and acceleration q had at earlier time, when the message left

Coulomb's Law

Force on a test charge Q due to a single point charge q at rest

$$\hookrightarrow \quad F = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r} \quad \text{---} (1)$$

ε_0 ⟶ Permittivity of free space.

$$\hookrightarrow 8.85 \times 10^{-12} \frac{C^2}{N\text{-}m^2}$$

$\mathfrak{r} \rightarrow$ separation vector

$\mathfrak{r} = r - r' \rightarrow$ location of q ---- (2)

$\downarrow$

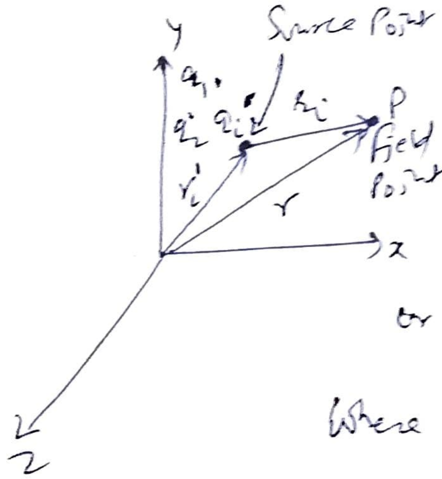
location of Q

Coulomb's Law & Superposition $\rightarrow$ physical input for electrostatics.

The Electric field

$q_1, q_2, q_3 \text{---} q_n$ at distances $\mathfrak{r}_1, \mathfrak{r}_2 \text{---} \mathfrak{r}_n$ from Q

$$F = F_1 + F_2 + \text{------} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{\mathfrak{r}_1^2}\hat{\mathfrak{r}}_1 + \frac{q_2 Q}{\mathfrak{r}_2^2}\hat{\mathfrak{r}}_2 + \text{---}\right)$$



$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1\hat{\mathfrak{r}}_1}{\mathfrak{r}_1^2} + \frac{q_2\hat{\mathfrak{r}}_2}{\mathfrak{r}_2^2} + \text{----}\right)$$

or $\Rightarrow$ $\boxed{F = QE}$ ---- (3)

Where $E = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathfrak{r}_i^2}\hat{\mathfrak{r}}_i$ ---- (4)

$\downarrow$

Electric field of Source charges

$\hookrightarrow$ Function of position (r),

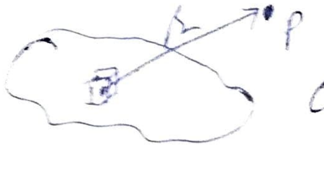
Separation vector $\mathfrak{r}_i$ $\rightarrow$ depends on the location of field point P.

$\hookrightarrow$ No reference to the test charge Q.

$E(r) \rightarrow$ Force per unit charge that would be exerted on a test charge, if you were to place one at P.

Continuous Charge Distributions

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathfrak{r}^2}\hat{\mathfrak{r}}\,dq \quad \text{---- (5)}$$

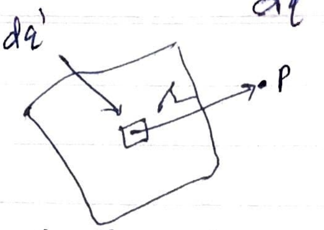


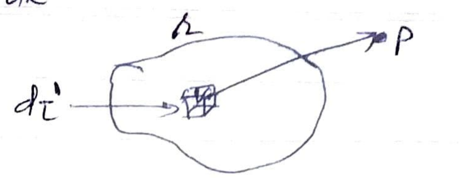
Continuous distribution

$dq = \lambda dl'$

line charge, λ ↳ charge per unit length

$$dq = \lambda\, dl'$$





Surface charge, σ

$$dq = \sigma\, da'$$

volume charge, ρ

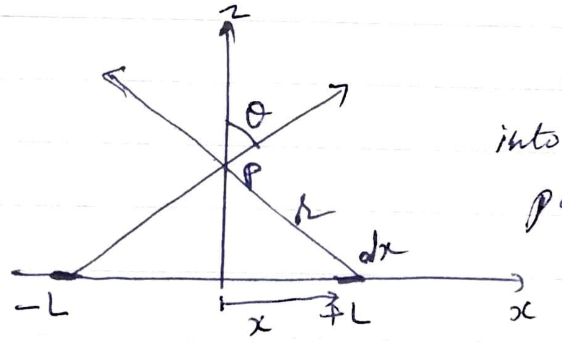
$$dq = \rho\, d\tau'$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int_P \frac{\lambda(r')}{\imath^2}\hat{\imath}\, dl' \quad --- (6)$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(r')}{\imath^2}\hat{\imath}\, da' \quad --- (7)$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(r')}{\imath^2}\hat{\imath}\, d\tau' \quad --- (8)$$

Prob. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform line charge λ



Chop the line up into symmetrically placed pairs (at $\pm x$)

↓

horizontal components of the two fields cancel

Net field of the pair is

$$dE = 2\,\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda dx}{\imath^2}\right)\cos\theta\,\hat{z}$$

$$\cos\theta = \frac{z}{\imath}, \quad \imath = \sqrt{z^2+x^2}, \quad x \text{ runs from } 0 \text{ to } L$$

$$E = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}\, dx$$

$$= \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^L$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$

For points far from the line ($z \gg L$)

$$E \cong \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$$

From far away the line "looks" like a point charge

$q = 2\lambda L$

In the limit $L \to \infty$, we obtain the field of an infinite straight wire

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$$

or, more generally

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s} \qquad (9)$$

$s \to$ distance from the wire

Divergence and Curl of Electrostatic Fields

Field Lines, Flux and Gauss's Law

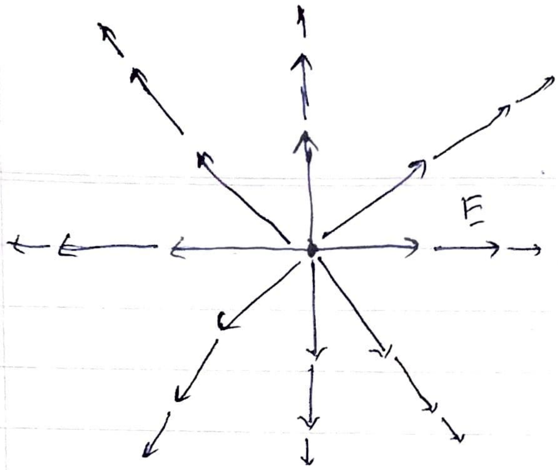
In principle $\to$ we are done with the subject of Electrostatics

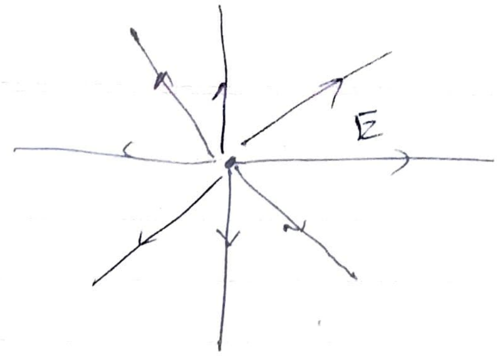
Integrals involved in computing E $\to$ can be formidable

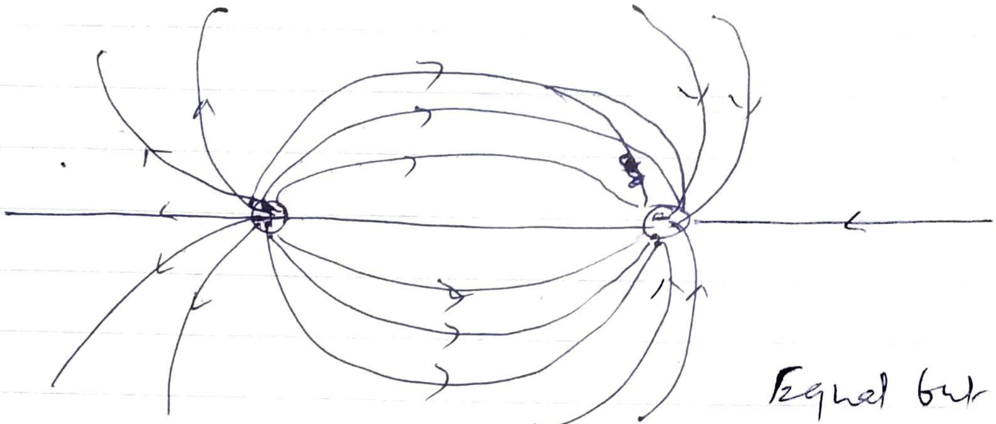
Rest of electrostatics $\to$ assembling a bag of tools and tricks for avoiding these integrals

A single point charge q, situated at the origin

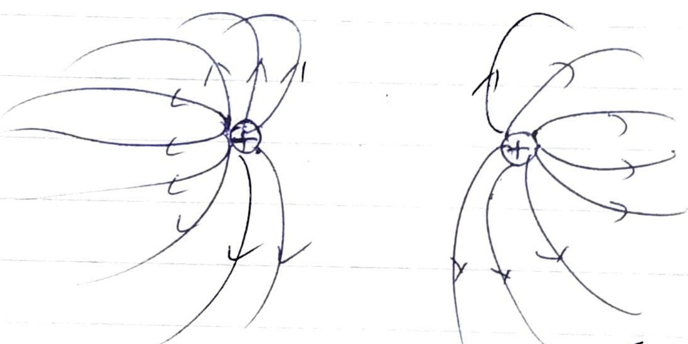
$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad (10)$$

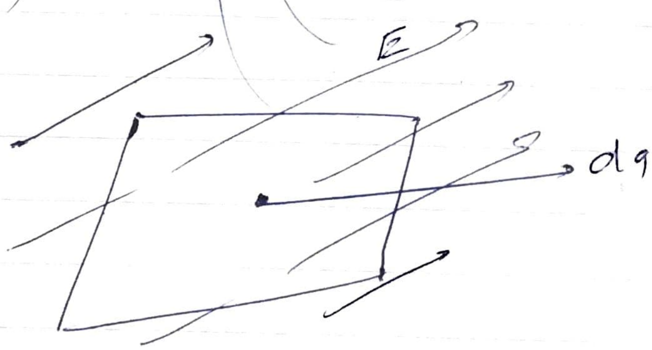






Equal but opposite charges



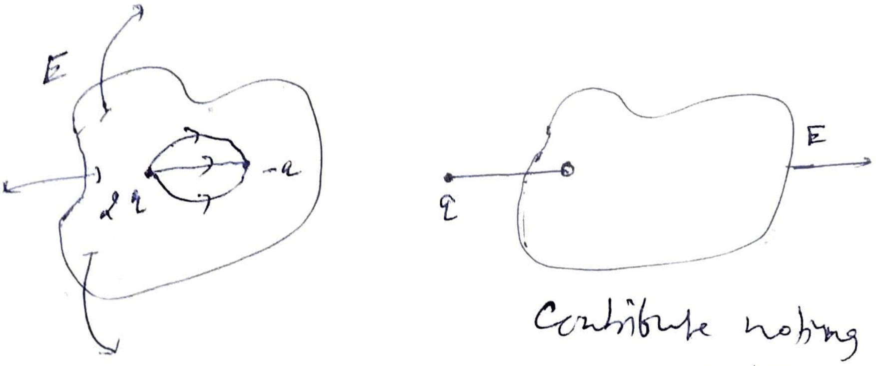


$$\phi_E \equiv \int_S E \cdot da \quad \text{--- (11)}$$

↳ measure of the "number of field lines" passing through S.

flux through any closed surface

↳ measure of the total charge inside



Contribute nothing to the flux

In case, a point charge q at the origin, the flux of E through a sphere of radius r, is

$$\oint E \cdot da = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \frac{1}{\varepsilon_0} q \quad \text{---} (12)$$

$$E = \sum_{i=1}^{n} E_i$$

The flux through a surface that encloses them all

$$\oint E \cdot da = \sum_{i=1}^{n}\left(\oint E_i \cdot da\right)$$

$$= \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0} q_i\right)$$

for any closed surface

$$\boxed{\oint_S E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}} \quad \text{---} (13)$$

From divergence theorem

$$\oint_S E \cdot da = \int_V (\nabla \cdot E)\, d\tau$$

$$Q_{enc} = \int_V \rho\, d\tau$$

Gauss's Law

$$\int_V (\nabla \cdot E)\, d\tau = \int_V \left(\frac{\rho}{\varepsilon_0}\right) d\tau \Rightarrow \boxed{\nabla \cdot E = \frac{1}{\varepsilon_0}\rho} \quad \text{---} (14)$$

The Divergence of E

from eqn (2-8)

$$E(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r')\, d\tau' \quad \text{—— (15)}$$

r-dependence is contained in $\mathscr{r} = r - r'$

$$\nabla . E = \frac{1}{4\pi\varepsilon_0} \int \nabla . \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \rho(r')\, d\tau'$$

$$\nabla . \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi \delta^3(\mathscr{r})$$

$$\Rightarrow \quad \nabla . E = \frac{1}{4\pi\varepsilon_0} \int 4\pi \delta^3(r - r') \rho(r')\, d\tau' = \frac{1}{\varepsilon_0} \rho(r)$$

—(16)

Applications of Gauss's Law

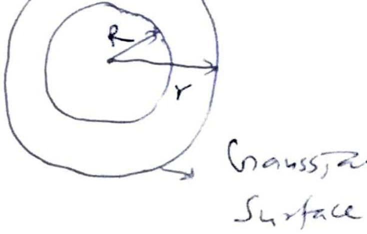
When symmetry permits → it affords the quickest and easiest way of computing electric fields.

Ex: Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Soln

Gaussian surface

$r > R$



$$\oint_S E . da = \frac{1}{\varepsilon_0} Q_{enc},$$

$$Q_{enc} = q$$

E → buried inside surface integral

Luckily, symmetry allows us to extract E from under the integral sign.

E → Points radially outward as does da

$$\Rightarrow \quad \oint_S E . da = \int_S |E|\, da$$

Magnitude of E → Constant over the Gaussian surface

$$\int_S |E|\, da = |E| \int_S da = |E|\, 4\pi r^2$$

Thus $$|E|\, 4\pi r^2 = \frac{1}{\varepsilon_0} q$$

or $$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}.$$

Field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center.

— x — x — x —

Gauss's Law → always true, but it is not always useful.

If ρ → not been uniform, or not spherically symmetrical

or Some other shape for → Gaussian surface

flux of E is $\left(\frac{1}{\varepsilon_0}\right) q$, but not certain that E was in same direction as da and constant in magnitude over the surface

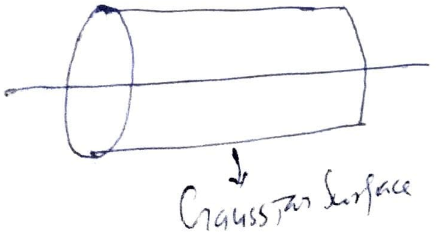
↓

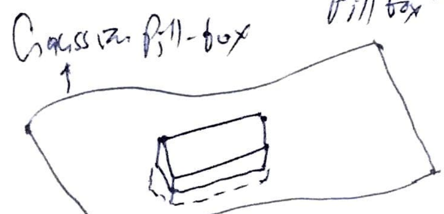
Without this, |E| can't be pulled out of Integral

Symmetry is crucial

Three kind of symmetry that works

1. Spherical symmetry → Gaussian surface a concentric sphere
2. Cylindrical symmetry → Gaussian surface a coaxial cylinder
3. Plane Symmetry → Gaussian "Pill box"





The Curl of E

A point charge at the origin

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}.$$

$\int_a^b E \cdot dl$ in spherical coordinates

$dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$, so

$$E \cdot dl = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

$$\Rightarrow \int_a^b E \cdot dl = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\Bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right) \quad \text{--- (17)}$$

The integral around a closed path is evidently zero (for then $r_a = r_b$)

$$\oint E \cdot dl = 0 \quad \text{--- (18)}$$

and hence, applying Stoke's theorem

$$\boxed{\nabla \times E = 0} \quad \text{--- (19)}$$

Eqn (18), (19) hold for any static charge distribution whatever.

Electric Potential

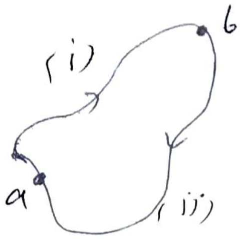
Electric field E → not just any vector function
→ very special kind of vector function
→ whose curl is always zero

This property of E → reduce a vector problem (finding E) down to a much simpler scalar problem

Any vector whose curl is zero → equal to the gradient of some scalar

We prove in ~~the~~ context of electrostatics

$\nabla \times E = 0$, $\longrightarrow$ the line integral of E around any closed loop is zero.



$\oint E \cdot dl = 0$, the integral of E from point 'a' to point 'b' is same for all paths.

Integral is independent of path, so

$$V(r) \equiv -\int_{O}^{r} E \cdot dl \quad \text{--- (20)}$$

$O \rightarrow$ reference point

$V \rightarrow$ only depends on the point r, $\rightarrow$ electric potential

$$V(b) - V(a) = -\int_{O}^{b} E \cdot dl + \int_{O}^{a} E \cdot dl$$

$$= -\int_{O}^{b} E \cdot dl - \int_{a}^{O} E \cdot dl = -\int_{a}^{b} E \cdot dl \quad \text{--- (21)}$$

Fundamental theorem for gradients

$$V(b) - V(a) = \int_{a}^{b} (\nabla V) \cdot dl$$

$$\Rightarrow \int_{a}^{b} (\nabla V) \cdot dl = -\int_{a}^{b} E \cdot dl$$

$$\Rightarrow \boxed{E = -\nabla V} \quad \text{--- (22)}$$

If the line integral of $E \rightarrow$ depends on path

the definition of $V \rightarrow$ eq (20) $\rightarrow$ nonsense

$\downarrow$

Changing the path would alter $V(r)$!

If we know V, we can easily get E.
Just take the gradient: $\underline{E = -\nabla V}$.

$E \rightarrow$ vector quantity (three components)
$V \rightarrow$ scalar (one component)

One function possibly contain all the information that "three" independent functions carry?

$E \rightarrow$ three components of E are not really independent
$\hookrightarrow$ They are explicitly interrelated, $\nabla \times E = 0$

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

Vector problem reduced to a scalar problem.

The reference point O

$$V'(r) = -\int_{O'}^{r} E \cdot dl = -\int_{O'}^{O} E \cdot dl - \int_{O}^{r} E \cdot dl$$

$$= k + V(r)$$

Adding a new constant k to $V \rightarrow$ will not affect the potential difference between two points

k cancel out $\leftarrow$ $V'(b) - V'(a) = V(b) - V(a)$

derivative of a constant zero and $\underline{\nabla V' = \nabla V}$

Potential obeys the superposition principle

$F = F_1 + F_2 + \ldots$

$E = E_1 + E_2 + \ldots$ Dividing through Q

$V = V_1 + V_2 + \ldots$ Integrating

Ex: Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge. Set the reference point at infinity.