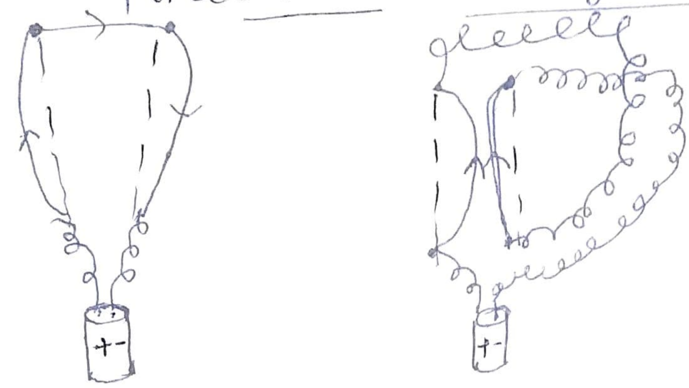
Magnetostatics

The Lorentz Force Law

Forces between charges in motion



Current in opposite directions repel

Current in same direction attract

Magnetic forces :- Magnetic force in a charge Q, moving with velocity v in a magnetic field B

$$F_{mag} = Q\,(v \times B) \quad \text{---(1)}$$

In presence of both electric & magnetic fields — Lorentz Force Law

$$F = Q\,[E + (v \times B)] \quad \text{---(2)}$$

$\boxed{\text{Magnetic forces do not work}}$

Direct Consequence of Lorentz Force Law

For if Q moves an amount $dl = v\,dt$, the work done is

$$dW_{mag} = F_{mag} \cdot dl$$

$$= Q\,(v \times B) \cdot v\,dt = 0 \quad \text{---(3)}$$

$(v \times B)$ is perpendicular to v

so $(v \times B) \cdot v = 0$

Magnetic forces → May alter the direction in which a particle moves, but they cannot speed it up or slow it down.

The Biot-Savart Law

Steady Currents

Continuous flow that has been going on forever without change

stationary charges → constant electric fields → Electrostatics

ρ needs to be stationary

Steady currents → constant magnetic fields → Magnetostatics

truly steady current
truly stationary charge] ?

→ Artificial worlds

→ They represent suitable approximation

↳ as long as the actual fluctuations are reasonably slow

↓

Magnetostatics → applies well to household currents

↓

which alternate 60 times a second!

Steady current in a wire → its magnitude I must be same all along the line

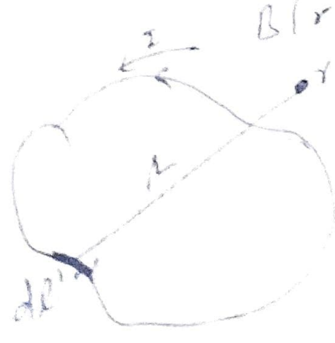
$\frac{\partial \rho}{\partial t} = 0$ in magnetostatics and hence the

Continuity eq^n becomes $\left[\nabla \cdot J = -\frac{\partial \rho}{\partial t}\right]$

$$\nabla \cdot J = 0 \quad \text{--- (1)}$$

The Magnetic field of a steady Current

Biot - Savart Law



$$B(r) = \frac{\mu_0}{4\pi} \int \frac{I \times \hat{r}}{r^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{dl' \times \hat{r}}{r^2} \quad \text{--- (2)}$$

$$\mu_0 = 4\pi \times 10^{-7} N/A^2 \quad \text{--- (3)}$$

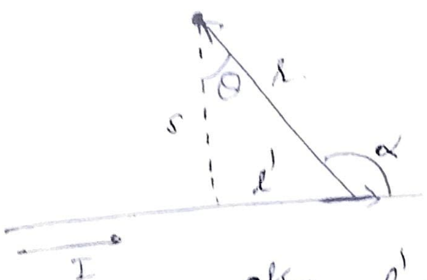
B → newtons per ampere-meter (required by the Lorentz force law) or Tesla T

$1\,T = 1\, N/A\,m$ ---(4)

Biot-Savart law → analogous to Coulomb's law in electrostatics

Common $\frac{1}{r^2}$ dependence

Prob: Find the magnetic field a distance s from a long straight wire carrying a steady current I



In the diagram, $\underrightarrow{dl' \times \hat{r}}$ points out of page, and has the magnitude

$$dl' \sin\alpha = dl' \cos\theta$$

also $l' = s\tan\theta$, so

$$dl' = \frac{s}{\cos^2\theta}\, d\theta$$

and $s = r\cos\theta$ $\quad \frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$

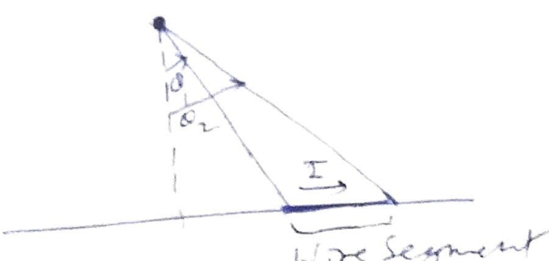
Thus

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1) \quad \text{---(5)}$$

Field of any straight segment of wire, in terms of the initial and final angles θ_1 and θ_2



A finite element by itself could never support a steady current → where would the charge go when it got to the end?

it might be piece of some closed circuit — eq (5) then represent its contribution to the total field.

In case of an infinite wire $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$

So

$$\boxed{B = \frac{\mu_0 I}{2\pi s}} \quad \text{---- (6)}$$

Force of attraction between two long, parallel wires a distance d apart carrying currents I_1 and I_2

The field at (2) due to (1) is

$$B = \frac{\mu_0 I_1}{2\pi d} \quad (7)$$ and it points into the edge

The Lorentz force law predicts a force directed towards (1) of magnitude

$$F = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl$$

Force per unit length is

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} \quad \text{-- (8)}$$

If the currents are antiparallel the force is repulsive

The divergence and curl of B

Straight line currents

[illegible]

Integral of B around a circular path of radius s centered at wire is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl$$

$$= \mu_0 I$$

↳ independent of s

B decreases at the same rate as the circumference increases.

Any loop that encloses the wire would give the same answer. If we use cylindrical coordinates (s, ϕ, z) with the current flowing along the z-axis

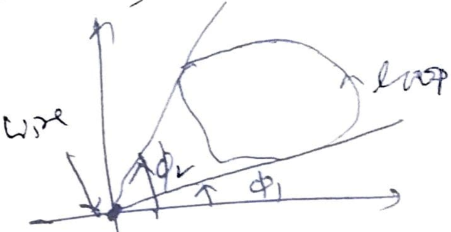
$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad \text{---} (9)$$

and $d\mathbf{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$ so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} s\,d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} 2\pi = \mu_0 I$$

Loop encloses wire exactly once.

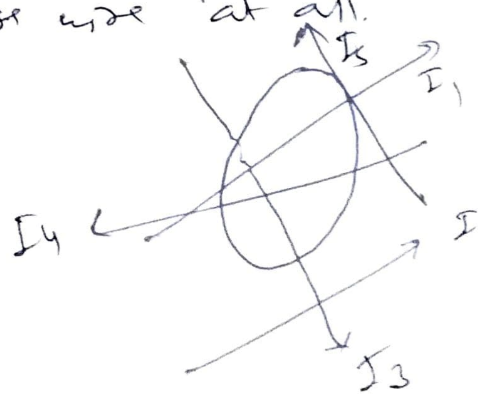
$$\int d\phi = 0$$



if loop didn't enclose wire at all.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \quad \text{---} (10)$$

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{a} \quad \text{---} (11)$$

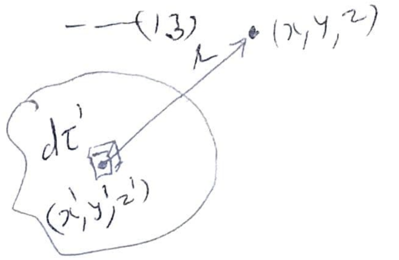


$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}} \quad \text{---} (12)$$

The Biot - Savart law for the general case of a volume current reads

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau' \quad \text{--- (13)}$$



Magnetic field at a point $\mathbf{r} = (x, y, z)$ in terms of an integral over the current distribution $\mathbf{J}(x', y', z')$

$\mathbf{B} \rightarrow$ function of (x, y, z)

$\mathbf{J} \rightarrow$ function of (x', y', z')

$\vec{\mathscr{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$

$d\tau' = dx'\, dy'\, dz'$

Integration $\rightarrow$ over the primed coordinates

Div. and Curl $\rightarrow$ w.r.t the <u>unprimed</u> coordinates

Applying the divergence to eq^n (13)

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau' \quad \text{--- (14)}$$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \quad \text{--- (15)}$$

$\nabla \times \mathbf{J} = 0 \longrightarrow \mathbf{J}$ does not depend on the unprimed variables (x, y, z)

$$\nabla \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 0$$

$$\Rightarrow \quad \boxed{\nabla \cdot \mathbf{B} = 0} \quad \text{--- (16)}$$

Applying curl to eq^n (13)

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau' \quad \text{--- (17)}$$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{--- (18)}$$

$$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\mathbf{r}) \quad \text{--- (19)}$$

$$\boxed{\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \, 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \, d\tau' = \mu_0 \mathbf{J}(\mathbf{r})}$$

Application of Ampere's law

$$\nabla \times B = \mu_0 J$$

$$\int (\nabla \times B)\cdot da = \oint B\cdot dl = \mu_0 \underbrace{\int J\cdot da}_{\text{Total current passing through the surface}}$$

Boundary line $\boxed{\oint B\cdot dl = \mu_0 I_{enc}}$

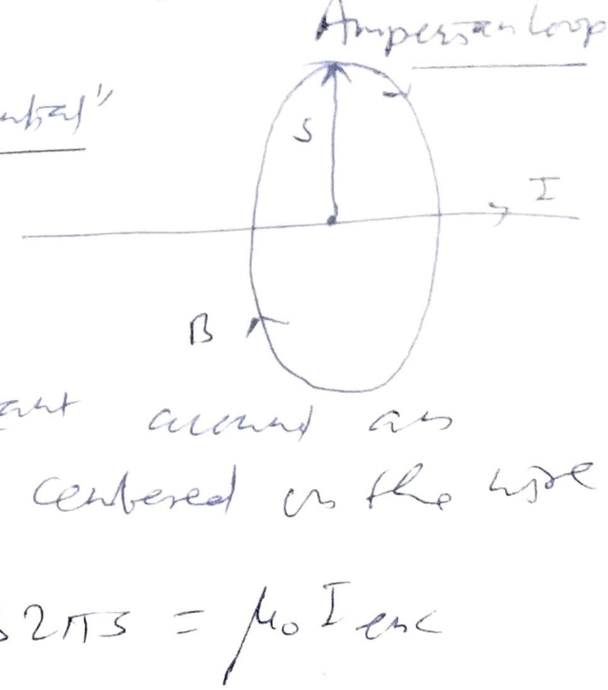
Electrostatics : Coulomb → Gauss
Magnetostatics : Biot-Savart → Ampere

for currents with appropriate symmetry → Ampere's Law

Ex: Find the magnetic field a distance s from a long straight wire, carrying a steady current I

Sol

Direction of B → "circumferential"
circling around the wire as indicate → right hand rule



By symmetry → B constant around an amperian loop of radius s, centered on the wire.

$$\oint B\cdot dl = B\oint dl = B\,2\pi s = \mu_0 I_{enc}$$
$$= \mu_0 I,$$

or $$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

Same answer as we obtained using Biot-Savart's Law but in less effort